Operators on some analytic function spaces and their dyadic counterparts

by

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Abstract

In this thesis we consider several questions on harmonic and analytic functions spaces and some of their operators. These questions deal with Carleson-type measures in the unit ball, bi-parameter paraproducts and multipliers problem on the bitorus, boundedness of the Bergman projection and analytic Besov spaces in tube domains over symmetric cones.

In part I of this thesis, we show how to generate Carleson measures from a class of weighted Carleson measures in the unit ball. The results are used to obtain boundedness criteria of the multiplication operators and Cesàro integral-type operators between weighted spaces of functions of bounded mean oscillation in the unit ball.

In part II of this thesis, we introduce a notion of functions of logarithmic oscillation on the bitorus. We prove using Cotlar's lemma that the dyadic version of the set of such functions is the exact range of symbols of bounded bi-parameter paraproducts on the space of functions of dyadic bounded mean oscillation. We also introduce the little space of functions of logarithmic mean oscillation in the same spirit as the little space of functions of bounded mean oscillation of Cotlar and Sadosky. We obtain that the intersection of these two spaces of functions of logarithmic mean oscillation and L^∞ is the set of multipliers of the space of functions of bounded mean oscillation in the bitorus.

In part III of this thesis, in the setting of the tube domains over irreducible symmetric cones, we prove that the Bergman projection P is bounded on the Lebesgue space L^p if and only if the natural mapping of the Bergman space $A^{p'}$ to the dual space $(A^p)^*$ of the Bergman space A^p, where $\frac{1}{p}+\frac{1}{p'}=1$, is onto. On the other hand, we prove that for $p>2$, the boundedness of the Bergman projection is also equivalent to the validity of an Hardy-type inequality. We then develop a theory of analytic Besov spaces in this setting defined by using the corresponding Hardy's inequality. We prove that these Besov spaces are the exact range of symbols of Schatten classes of Hankel operators on the Bergman space A^2.

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Statement

This thesis is submitted in accordance with the regulations for the degree of Doctor of Philosophy at the University of Glasgow.

Chapter 1 cover some basic properties of the unit ball. They are taken essentially from [117]. Chapter 5 is an introduction to the Analysis of symmetric cones and it follows the lines of the book [40].

Chapter 2 except Theorem 2.2.17, Chapter 6, Chapter 8, section 3.1.3 and 7.4.7 are original work of the author with the exception of instances indicated within the text.

Chapter 3 and 4 are from the joint works with Doctor S. Pott [85] and [86]. Theorem 2.2.17 is from the published joint work with Professors A. Bonami and S. Grellier [23]. Chapter 7 except section 7.4.7 is from the joint work with Professors D. Békollé, A. Bonami, G. Garrigös and F. Ricci [15]; a work to appear.

Chapter 2 except Theorem 2.2.17 is from the work to appear [95]. Chapter 6 is from the published work [93]; Chapter 8 is from the published work [94]. Lemma 3.2.5 is from the preprint [96].

Introduction

In this thesis, we are concerned with several problems related to the boundedness of various linear operators on some harmonic and holomorphic function spaces. We partially consider problems of the same type related to holomorphic function spaces in domains with different geometric structures. If in the unit ball (Part I) all the ingredients of real analysis are available, the case of the bidisc required the development of more elaborated techniques (see Part II). The case of tubular domains over irreducible symmetric cones of rank greater than 1 (see Part II) is even much more difficult and its study has been at the origin of some famous counter-examples such as the fact that the characteristic function in the disk is not a Fourier multiplier for Lebesgue spaces of exponent not equal to 2 [42] (this allows to prove that the Szegö projection is not bounded on L^p for $p \neq 2$) or some counter-examples to maximal inequalities proved using Kakeya sets (see the survey paper [70]).

0.1 Carleson-type measures in the unit ball of $\mathbb{C}^n$

Let Ω be a region in $\mathbb{C}^n$ and X a Banach space of continuous functions in Ω. A positive measure μ in Ω is called a p-Carleson measure for X, if there exists a positive constant $C = C(\mu)$ with the property that

$$\int_\Omega |f(z)|^p d\mu(z) \leq C\|f\|_X^p$$

for all $f \in X$.

For Y another Banach space of continuous functions in Ω, we pose the following problem: Characterize those positive measures μ in Ω such that given $0 \leq q < \infty$, the measures $|f(z)|^q d\mu(z)$ are p-Carleson measures for X for all $f \in Y$. In the first part of this thesis, we study such measures when Y is the space of functions of bounded mean oscillation or of logarithmic mean ascillation (respectively the Bloch space or the logarithmic Bloch space) and X is a Hardy space (respectively a weighted Bergman space).

0.1.1 Overview and motivation

When $q = 0$ and $Y = X = \mathcal{H}^p$ where $\mathcal{H}^p$ is the usual Hardy space, these measures are known as Carleson measures. Carleson [27] was the first to study such measures in the case of the unit disc $\mathbb{D}$ of $\mathbb{C}$. The Hardy space $\mathcal{H}^p(\mathbb{D})$ consists of holomorphic functions f in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with the property that:

$$\sup_{0 \le r < 1} \int_{-\pi}^{\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

Carleson measures have been shown to be very useful in many questions in Analysis. One question where they appear is the problem of pointwise multipliers of function spaces. i.e. the characterization of those functions f for which the mapping :

$$M_f : X \to X$$

$$g \mapsto f \cdot g$$

is a continuous mapping in the Banach space X. We can mention as well their role in the solution of various questions such as the Corona problem [27] and the characterization of the dual space of $\mathcal{H}^1$ [41,44]. Carleson measures also play an important role in interpolation problems with application to control theory (see e.g. [63]).

The extension of Carleson measures (for Hardy spaces) to the unit ball of $\mathbb{C}^n$

$$\mathbb{B}^n = \{z \in \mathbb{C}^n : |z| < 1\}$$

is due to Hörmander [60]. In 1982, Cima and Wogen [32] characterized Carleson measures for weighted Bergman spaces $A^p_\alpha(\mathbb{B}^n)$ $(\alpha > -1)$ in the unit ball of $\mathbb{C}^n$. The weighted Bergman space A^p_α $(\alpha > -1)$ consists of holomorphic functions f in $\mathbb{B}^n$ such that

$$\int_{\mathbb{B}^n} |f(z)|^p (1 - |z|^2)^\alpha dV(z) < \infty.$$

The space of functions of bounded mean oscillation in the unit disc $\mathbb{D}$ of $\mathbb{C}$, denoted $BMOA$, consists of holomorphic functions f in $\mathbb{D}$ such that

$$\sup_{I \subset \mathbb{T}} \frac{1}{|I|} \int_I |f(t) - m_I f| dt < \infty. \tag{0.1.1}$$

Here $\mathbb{T}$ is the unit circle, I an interval in $\mathbb{T}$ with length $|I|$ and $m_I f = \frac{1}{|I|} \int_I f(t) dt$ is the mean of f over I. The Bloch space $\mathcal{B}$ of the unit disc consists of the holomorphic functions f in $\mathbb{D}$ such that

$$\sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)| < \infty.$$

In 2003, Ruhan Zhao [111] gave a characterization in the unit disc of those measures μ such that $|f(z)|^p d\mu(z)$ is a Carleson measure for Hardy space (respectively for weighted Bergman spaces) for all $f \in BMOA$ (respectively $f \in \mathcal{B}$).

Given an interval I, we denote by $|I|$ the normalized length of I so that $|\partial\mathbb{D}| = |\mathbb{T}| = 1$. Let $S(I)$ be the subset of $\mathbb{T}$ defined as follows

$$S(I) = \{z \in \mathbb{C} : 1 - |I| < |z| < 1, \ \frac{z}{|z|} \in I\}.$$

The above set is also called Carleson square or region. R. Zhao [111] proved that the measure $|f(z)|^p d\mu(z)$ is a Carleson measure for Hardy spaces (respectively, for the weighted Bergman spaces $A^q_{s-2}(\mathbb{D})$, $s > 1$) for any $f \in BMOA$ (respectively $f \in \mathcal{B}$) if and only if there exists a constant $C > 0$ such that for any arc $I \subset \mathbb{T}$,

$$\mu(S(I)) \leq C \frac{|I|}{(\log \frac{2}{|I|})^p}$$

(respectively $\mu(S(I)) \leq C \frac{|I|^s}{(\log \frac{2}{|I|})^p}$). As applications of his result, he was able to characterize pointwise mutipliers of $BMOA$ and $\mathcal{B}$ in terms of these measures. He also obtained boundedness conditions of integral operator J_f with holomorphic symbol f defined on holomorphic functions by

$$J_f g(z) = \int_0^z g(\zeta) f'(\zeta) d\zeta. \tag{0.1.2}$$

0.1.2 On some equivalent definitions of ρ- Carleson measures on the unit ball

Recall that for $\alpha > -1$ the weighted Lebesgue measure dV_α is defined by

$$dV_\alpha(z) = c_\alpha (1 - |z|^2)^\alpha dV(z), \tag{0.1.3}$$

where

$$c_\alpha = \frac{\Gamma(n + \alpha + 1)}{n!\Gamma(\alpha + 1)} \tag{0.1.4}$$

is a normalizing constant so that $V_\alpha(\mathbb{B}^n) = 1$.

Definition 0.1.1. For $\alpha > -1$ and $0 < p < \infty$, the weighted Bergman space $A^p_\alpha(\mathbb{B}^n)$ consists of holomorphic functions f in $L^p(\mathbb{B}^n, dV_\alpha)$, that is

$$A^p_\alpha(\mathbb{B}^n) = L^p(\mathbb{B}^n, dV_\alpha) \bigcap H(\mathbb{B}^n). \tag{0.1.5}$$

We use the notation

$$||f||_{p,\alpha}^p := \int_{\mathbb{B}^n} |f(z)|^p dV_\alpha(z) \tag{0.1.6}$$

for $f \in L^p(\mathbb{B}^n, dV_\alpha)$.

Definition 0.1.2. For $0 < p < \infty$ the Hardy space $\mathcal{H}^p(\mathbb{B}^n)$ is the space of all $f \in H(\mathbb{B}^n)$ such that

$$||f||_p^p := \sup_{0<r<1} \int_{\mathbb{S}^n} |f(r\xi)|^p d\sigma(\xi) < \infty. \tag{0.1.7}$$

The space of all bounded holomorphic functions in $\mathbb{B}^n$ will be denoted $\mathcal{H}^\infty(\mathbb{B}^n)$.

For any $\xi \in \mathbb{S}^n$ and $\delta > 0$, let

$$B_\delta(\xi) = \{w \in \mathbb{S}^n : |1 - \langle w, \xi\rangle| < \delta\},$$

and

$$Q_\delta(\xi) = \{z \in \mathbb{B}^n : |1 - \langle z, \xi\rangle| < \delta\}.$$

These last ones are the higher dimension analogues of Carleson regions. For $f \in \mathcal{H}^1(\mathbb{B}^n)$, we still denote by $f(\xi)$, for $\xi \in \mathbb{S}^n$, the admissible limit at the boundary, which exists a.e. The space of functions of bounded mean oscillation $BMOA$ is the space of all $f \in \mathcal{H}^1(\mathbb{B}^n)$ for which there exists a constant $C > 0$ so that

$$\sup_{\substack{B=B_\delta(\xi),\\ \delta\in]0,1[,\xi\in\mathbb{S}^n}} \frac{1}{\sigma(B)} \int_B |f - f_B| d\sigma \leq C.$$

Here and anywhere else, f_B denotes the mean value of f on B.

The space $BMOA$ is a Banach space when equipped with the norm

$$||f||_{BMOA} = |f(0)| + \sup_{\substack{B=B_\delta(\xi),\\ \delta\in]0,1[,\xi\in\mathbb{S}^n}} \frac{1}{\sigma(B)} \int_B |f - f_B| d\sigma.$$

We now define the space of functions of logarithmic mean oscillation $LMOA$. An analytic function f belongs to $LMOA$ if $f \in \mathcal{H}^1(\mathbb{B}^n)$ and there exists a constant $C > 0$ so that

$$\sup_{\substack{B=B_\delta(\xi),\\ \delta\in]0,1[,\xi\in\mathbb{S}^n}} \frac{\log \frac{4}{\delta}}{\sigma(B)} \int_B |f - f_B| d\sigma \leq C.$$

The space $LMOA$ is a Banach space when equipped with the norm

$$||f||_{LMOA} = |f(0)| + \sup_{\substack{B=B_\delta(\xi),\\ \delta\in]0,1[,\xi\in\mathbb{S}^n}} \frac{\log \frac{4}{\delta}}{\sigma(B)} \int_B |f - f_B| d\sigma.$$

The radial derivative Rf of a holomorphic function f is given by

$$Rf(z) = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z).$$

The Bloch space $\mathcal{B}$ consists of all $f \in H(\mathbb{B}^n)$ such that

$$||f||_{\mathcal{B}} = |f(0)| + \sup_{z \in B_n} |Rf(z)|(1 - |z|^2) < \infty. \tag{0.1.8}$$

We also recall the following definition of the logarithmic (weighted) Bloch space $L\mathcal{B}$. An analytic function f belongs to $L\mathcal{B}$ if

$$\sup_{z \in B_n} |Rf(z)|(1 - |z|^2) \log \frac{4}{1 - |z|^2} < \infty. \tag{0.1.9}$$

The natural norm on $L\mathcal{B}(\mathbb{B}^n)$ is given by

$$||f||_{L\mathcal{B}} = |f(0)| + \sup_{z \in B_n} |Rf(z)|(1 - |z|^2) \log \frac{4}{1 - |z|^2} < \infty. \tag{0.1.10}$$

Both $\mathcal{B}$ and $L\mathcal{B}$ are also Banach when equipped with the norms $|| \cdot ||_{\mathcal{B}}$ and $|| \cdot ||_{L\mathcal{B}}$, respectively.

Now we consider generalized Carleson type measures with additional logarithmic terms.

Definition 0.1.3. Let μ be a positive Borel measure on $\mathbb{B}^n$ and $0 < s < \infty$. For ρ a positive function defined on $(0, 1)$, we say μ is a (ρ, s)- Carleson measure if there is a constant $C > 0$ such that for any $\xi \in \mathbb{S}^n$ and $0 < \delta < 1$,

$$\mu(Q_\delta(\xi)) \le C \frac{(\sigma(B_\delta(\xi)))^s}{\rho(\delta)}. \tag{0.1.11}$$

When $s = 1$, μ is called a ρ-Carleson measure and if moreover ρ is a constant function, then such measures are exactly Carleson measures. In Chapter 2, we consider the particular case $\rho(t) = \rho_{p,q}(t) = (\log(4/t))^p(\log\log(e^4/t))^q$ with $0 \le p, q < \infty$. As seen above,the case $\rho(t) = (\log(4/t))^p$ has been studied in [111]. To characterize such measures, we adapt and extend ideas of [111] to the unit ball of $\mathbb{C}^n$. Our main results are a series of criteria of these measures. In particular, we prove:

Theorem 0.1.4. *Let $0 \le p, q < \infty$ and let μ be a positive Borel measure on $\mathbb{B}^n$. Then the following conditions are equivalent.*

i) There is $C_1 > 0$ such that for any $\xi \in \mathbb{S}^n$ and $0 < \delta < 1$,

$$\mu(Q_\delta(\xi)) \le C_1 \frac{\sigma(B_\delta(\xi))}{(\log \frac{4}{\delta})^p (\log\log \frac{e^4}{\delta})^q}.$$

ii) For any $f \in BMOA$ and any $g \in LMOA$, the measure $|f(z)|^p|g(z)|^q d\mu(z)$ is a Carleson measure.

Theorem 0.1.5. *Let $0 \le p, q < \infty$, $1 < s < \infty$. Let μ be a positive Borel measure on $\mathbb{B}^n$. Then the following conditions are equivalent.*

i) There is $C_1 > 0$ such that for any $\xi \in \mathbb{S}^n$ and $0 < \delta < 1$,

$$\mu(Q_\delta(\xi)) \le C_1 \frac{(\sigma(B_\delta(\xi)))^s}{(\log \frac{4}{\delta})^p (\log\log \frac{e^4}{\delta})^q}.$$

ii) For $0 < r < \infty$ and for any $f \in \mathcal{B}$ and any $g \in L\mathcal{B}$,the measure $|f(z)|^p|g(z)|^q d\mu(z)$ is a Carleson measure for $A^r_{ns-(n+1)}(\mathbb{B}^n)$.

Among the consequences of our characterization, are boundedness criteria of the higher dimensional version of the operator (0.1.2) on $LMOA$ and between $LMOA$ and $BMOA$. It is a result of D. Stegenga [100] and independently of S. Janson [64] that the space of pointwise multipliers of $BMOA$ in the unit disc of $\mathbb{C}$ is exactly the intersection $L^\infty \cap LMOA$. This result has been extended to weighted $BMOA$ by Janson [64]. With our characterization, we also describe the set of pointwise multipliers of $LMOA$ and from this space to $BMOA$, extending partially to the unit ball the results of [64].

0.2 Logarithmic mean oscillation in the bidisc

Given $z = re^{i\theta} \in \mathbb{D}$, let us denote by I_z the arc

$$I_z = \{e^{i\omega} : |\omega - \theta| < 1 - r\}.$$

The Carleson squares defined in the previous section can be also written as follows

$$S(I) = \{z \in \mathbb{D} : I_z \subset I\}.$$

Given $f \in L^2(\mathbb{T})$ ($\mathbb{T}$ is the unit circle), we denote by F its Poisson extension to $\mathbb{D}$. It is well-known that a measure μ is such that there exists a constant $C > 0$ with

$$\int_{\mathbb{D}} |F(z)|^2 d\mu(z) \le C \int_{\mathbb{T}} |f(e^{i\theta})|^2 d\theta \quad \text{for all} \quad f \in L^2(\mathbb{T}) \tag{0.2.1}$$

if and only if there is a constant K with

$$\mu(S(I)) \le K|I|$$

for all intervals I. One could expect that in two dimension, the Carleson Embedding (0.2.1) holds exactly for those measures μ in $\mathbb{D}\times\mathbb{D}$ for which there exists a constant $C>0$ such that

$$\mu(S(I\times J))\leq C|I||J|$$

for all product of Carleson squares $S(I\times J)=S(I)\times S(J)$ over rectangles $I\times J$. In fact this is not the case, as proved by Carleson by constructing a counter-example [28].

In 1979, S-Y. A. Chang proved in [29] that a measure μ satisfies the embedding (0.2.1) in the case of the bidisc $\mathbb{D}^2$ if and only if there exists a constant $C>0$ such that

$$S(\Omega)\leq C|\Omega| \tag{0.2.2}$$

for all connected open set $\Omega\subset\mathbb{T}^2$, where

$$S(\Omega)=\{(z_1,z_2)\in\mathbb{D}^2:\, S(I_{z_1})\times S(I_{z_2})\subset\Omega.\}$$

This show how far the situation in the product spaces can be different and even complicated. In 1980, S-Y. A. Chang and R. Fefferman [30] characterized the dual space of the real Hardy space H^1_{Re} of the bidisc defined by

$$H^1_{Re}(\mathbb{T}^2)=\{f\in L^1: H_1f\in L^1, H_2f\in L^1, H_1H_2f\in L^1\},$$

where H_1 and H_2 are Hilbert transforms in the first and the second variable, respectively and the Hilbert transform is defined on $L^1(\mathbb{T})$ by

$$Hf(x):=\text{p.v.}\frac{1}{\pi}\int_0^1\frac{f(y)}{\tan(\pi(x-y))}dy. \tag{0.2.3}$$

They proved that this dual space is the space of functions of bounded mean oscillation BMO corresponding to the characterization (0.2.2) of Carleson measures in the bidisc in the following sense: A function f belongs to $BMO(\mathbb{T}^2)$ if and only if the measure $|\nabla F(z_1,z_2)|^2\log\frac{1}{z_1}\log\frac{1}{z_2}dz_1d\overline{z_1}dz_2d\overline{z_2}$ is a Carleson measure on the bidisc, F being the bi-harmonic extension of f to $\mathbb{D}^2$ and

$$|\nabla F(z_1,z_2)|^2=\left(\left|\frac{\partial^2F}{\partial z_1\partial z_2}\right|^2+\left|\frac{\partial^2F}{\partial z_1\partial\overline{z_2}}\right|^2+\left|\frac{\partial^2F}{\partial\overline{z_1}\partial z_2}\right|^2+\left|\frac{\partial^2F}{\partial\overline{z_1}\partial\overline{z_2}}\right|^2\right)(z_1,z_2).$$

The Chang-Fefferman $BMO(\mathbb{T}^2)$ space has been the subject of several works. Its dyadic version for example is the right range of symbol of some bounded paraproducts between Lebesgue spaces in product domains [21, 71, 84].

Pointwise multipliers of the space of functions of bounded mean oscillation $BMO(\mathbb{T})$ are well understood. In [100], D. Stegenga proved that they correspond exactly to those bounded functions with logarithmic mean oscillation, that is, $L^\infty \cap LMO$ is exactly the algebra of functions f such that the multiplication operator by f is bounded on $BMO(\mathbb{T})$. Thus, it is a natural question to ask what the pointwise multipliers of $BMO(\mathbb{T}^2)$ are.

Our approach uses essentially dyadic techniques. Let us identify the unit circle $\mathbb{T}$ with the interval $[0,1]$. A dyadic interval is any interval of the form $[k2^{-j},(k+1)2^{-j})$ with j,k integers. Let h_I denote the Haar wavelet adapted to the dyadic interval I,

$$h_I = |I|^{-1/2}(\chi_{I^+} - \chi_{I^-}),$$

where I^+ and I^- are the right and left halves of I, respectively and χ_I is the characteristic function of I. The set of functions $\{h_I : I \in \mathcal{D}\} \cup \{\chi_{[0,1]}\}$ forms an orthonormal basis for $L^2([0,1])$ (see [81]). We denote by $\mathcal{D}$ the set of dyadic intervals in $\mathbb{T}$ and we denote by $\mathcal{R}$ the set of all dyadic rectangles $R = I \times J$, I and J in $\mathcal{D}$. The product Haar wavelet adapted to $R = I \times J$ is defined by $h_R(t,s) = h_I(t)h_J(s)$. For $f \in L^2(\mathbb{T}^2)$ with mean zero over $\mathbb{T}^2$ we have the representation:

$$f = \sum_{R\in\mathcal{R}} \langle f, h_R\rangle h_R = \sum_{R\in\mathcal{R}} f_R h_R.$$

We will be writing $m_R f$ for the mean of $f \in L^2(\mathbb{T}^2)$ over the dyadic rectangle R.

The space of functions of dyadic bounded mean oscillation in $\mathbb{T}^2$, $\mathrm{BMO}^d(\mathbb{T}^2)$, is the space of all function $f \in L^2(\mathbb{T}^2)$ such that

$$||f||^2_{BMO^d} := \sup_{\Omega\subset\mathbb{T}^2} \frac{1}{|\Omega|} \sum_{R\in\Omega} |f_R|^2 = \sup_{\Omega\subset\mathbb{T}^2} \frac{1}{|\Omega|} ||P_\Omega f||_2^2 < \infty, \tag{0.2.4}$$

where the supremum is taken over all open sets $\Omega \subset \mathbb{T}^2$ and P_Ω the orthogonal projection on the subspace spanned by Haar functions h_R, $R \in \mathcal{R}$ and $R \in \Omega$. It is well-known (see [30]) that $\mathrm{BMO}^d(\mathbb{T}^2)$ is the dual space of the dyadic product Hardy space $H^1_d(\mathbb{T}^2)$ defined in terms of the dyadic square function

$$\mathcal{S}(f)(t,s) = (\sum_{(t,s)\in R\in\mathcal{R}} \frac{|\langle f, h_R\rangle|^2}{|R|})^{1/2}.$$

That is,

$$H^1_d(\mathbb{T}^2) = \{f \in L^1(\mathbb{T}^2) : \mathcal{S}f \in L^1(\mathbb{T}^2)\}.$$

Given two function f and g in $L^2(\mathbb{T}^2)$ with finite Haar expansion, the pointwise product $f \cdot g$ can be written as the following

$$fg = \pi_{\pi_g} f + \Delta_g f + \pi_{\Delta_g} f + \Delta_{\pi_g} f + R_{\Delta_g} f + \Delta_{R_g} f + R_{\pi_g} f + \pi_{R_g} f + R_{R_g} f. \tag{0.2.5}$$

The nine terms correspond to the matrix elements $\langle M_\varphi h_I(s)h_J(t), h_{I'}(s)h_{J'}(t)\rangle$ for $I' \subset I$, $I' = I$, $I' \subset I$, $I' \supset I$, $J' \subset J$, $J' = J$, $J' \supset J$. The first four operators above are non diagonal terms of the matrix, we call them "paraproducts".

Remark 0.2.1. In one dimension, for f and b with finite Haar expansion, we have

$$fb = \pi_b(f) + (\pi_{\overline{b}})^*(f) + \pi_f(b),$$

where π_b is the dyadic paraproduct with symbol b defined on $L^2(\mathbb{T})$ by

$$\pi_b(f) = \sum_{I\in\mathcal{D}} b_I m_I f h_I$$

and its adjoint $(\pi_b)^* = \Delta_{\overline{b}}$ is given by

$$\Delta_b(f) = \sum_{I\in\mathcal{D}} b_I f_I \frac{\chi_I}{|I|}.$$

Thus the terms in (0.2.5) can be viewed as composition of the one dimensional operators π, Δ and R. We have in particular

$$\pi_{\pi_\varphi}(f) = \sum_{R\in\mathcal{R}} h_R \varphi_R m_R f.$$

Continuous versions of Paraproducts first appeared in the work of Bony [26] in relation with non linear differential equations. Since then they have appeared as important tool in Harmonic Analysis and have been extensively studied [21, 54, 71–73, 75, 76, 78, 87]. Their importance can be illustrated from the $T(1)$ theorem of David and Journé [69] which claims that many singular integral operators T can be written as $T = S + \pi_b + (\pi_b)^*$, where S is an almost translation invariant (or convolution) operator.

In Chapter 3, we study the boundedness of the four paraproducts in (0.2.5) on $BMO^d(\mathbb{T}^2)$. For this, we introduce some notions of functions of logarithmic mean oscillation in $\mathbb{T}^2$ that generalize the one dimensional one.

Definition 0.2.2. Let $\varphi \in L^2(\mathbb{T}^2)$. Then $\varphi \in \mathrm{LMO}^d(\mathbb{T}^2)$, if and only if there exists $C > 0$ such that for each dyadic rectangle $R = I \times J$ and each open set $\Omega \subseteq R$,

$$\frac{\log(\frac{4}{|I|})^2 \log(\frac{4}{|J|})^2}{|\Omega|} \sum_{Q\in\mathcal{R}, Q\subseteq\Omega} |\varphi_Q|^2 \leq C.$$

As a key result, we obtain the following.

Theorem 0.2.3. *Let $\varphi \in L^2(\mathbb{T}^2)$. Then $\varphi \in \mathrm{LMO}^d(\mathbb{T}^2)$, if and only if $\pi_{\pi_\varphi} : \mathrm{BMO}^d(\mathbb{T}^2) \to \mathrm{BMO}^d(\mathbb{T}^2)$ is bounded, and $\|\pi_{\pi_\varphi}\|_{\mathrm{BMO}^d\to\mathrm{BMO}^d} \approx \|\varphi\|_{\mathrm{LMO}^d}$.*

Given two positive quantities A and B which depend on parameters $\alpha_1, \alpha_2, \cdots$, the notation $A \approx B$ means that

$$cB \leq A \leq CB$$

where c and C are independent of some or all of the parameters $\alpha_1, \alpha_2, \cdots$. Although not normally stated explicitly, it is usually clear from the context which parameters c and C are independent of. There are analogous definitions for the notation $\sim$, $\lesssim$ and $\gtrsim$ which are used later. The proofs in this chapter use a decomposition of the operators as a sum of localized operators satisfying some good estimates, and Cotlar's lemma.

In Chapter 4, using the results of Chapter 3, we characterize the algebra of pointwise multipliers of $BMO(\mathbb{T}^2)$. Let $\mathrm{LMO}(\mathbb{T}^2)$ be the intersection of all dyadic $\mathrm{LMO}(\mathbb{T}^2)$ obtained by translating the original dyadic grid $\mathcal{R}$, and let $\mathrm{lmo}(\mathbb{T}^2)$ be the set defined as follows:

$$\begin{aligned}\mathrm{lmo}(\mathbb{T}^2) = \{b \in L^2(\mathbb{T}^2) : \exists C > 0 \text{ such that } &\|m_I^{(1)} b\|_{\mathrm{LMO}(\mathbb{T})} \leq C,\\ &\|m_J^{(2)} b\|_{\mathrm{LMO}(\mathbb{T})} \leq C \text{ for all intervals } I, J \subset \mathbb{T}\}.\end{aligned}$$

We prove exactly the following result.

Theorem 0.2.4. *The set of pointwise multipliers of $BMO(\mathbb{T}^2)$ is the intersection* $\mathrm{lmo}(\mathbb{T}^2) \cap \mathrm{LMO}(\mathbb{T}^2) \cap L^\infty(\mathbb{T}^2)$.

0.3 Hardy-type inequalities and Analytic Besov spaces in tubular domains over symmetric cones

Let D be a domain in $\mathbb{C}^n$ and denote by dV the Lebesgue measure in $\mathbb{C}^n$. The Bergman space $A^2(D)$ is the closed subspace of the Lebesgue space $L^2(D, dV)$ consisting of holomorphic functions in D. Let us denote by P the orthogonal projection from $L^2(D, dV)$ onto $A^2(D)$. P is given by

$$Pf(z) = \int_D B(z,w) f(w) dV(w), \quad f \in L^2(D, dV) \tag{0.3.1}$$

where $B(.,.)$ is the Bergman kernel of D.

A main concern in the kind of analysis we are interested in is the characterization of those $p \in [1, \infty[$ for which P extends as a bounded operator from $L^p(D, dV)$ into itself. The answer to this question is completely known in one dimension. Indeed, in the case of unit disc $\mathbb{D}$ or the upper-half plane $\mathbb{C}_+$ of $\mathbb{C}$, it is well-known that P is bounded on

$L^p(D, dV)$ if and only if $1 < p < \infty$ (see [12, 59, 88]). The one dimensional result has been extended to the unit ball of $\mathbb{C}^n$ by F. Forelli and W. Rudin [48] in 1974. For general domains, the question is still open. The case of pseudo-convex and bounded domains of $\mathbb{C}^n$ has been considered in [39, 43, 82]. Partial results have been obtained by various authors in various settings specially for Siegel domains of type II and tubular domains over homogeneous cones [9, 10, 12, 14, 17, 18, 38, 56, 77].

The interest of the above question can be illustrated by some of its consequences among which is the fact that if P extends as a bounded operator on $L^p(D, dV)$, $p > 1$, then the dual space of the Bergman space $A^p(D)$ identifies with the Bergman space $A^{p'}(D)$ where $\frac{1}{p} + \frac{1}{p'} = 1$. Consequently, denoting by $(A^p(D))^*$ the dual of the Bergman space $A^p(D)$, we raise the following question: Is there any equivalence between the boundedness of P and the surjectivity of the natural mapping between $(A^p(D))^*$ and $A^{p'}$?

In one dimension, it is well-known that a function f belongs to the Bergman space A^p if and only if the function $(1 - |z|^2)f'(z)$ is in $L^p(\mathbb{D}, dV)$ and there are constants $c, C > 0$ such that

$$c \int_{\mathbb{D}} |f(z)|^p dV(z) \leq \int_{\mathbb{D}} \left[(1 - |z|^2)|f'(z)|\right]^p dV(z) \leq C \int_{\mathbb{D}} |f(z)|^p dV(z). \qquad (0.3.2)$$

If the second inequality can be extended to higher dimension using the mean value inequality, the first one (Hardy inequality) is not so natural for all exponents p and for more general domains in $\mathbb{C}^n$. Thus, we pose our second question, which is to know if the boundedness of P is equivalent to the validity of the corresponding Hardy inequality in such domains.

Note that if the equivalent formulation in the domain D of the first inequality in (0.3.2) does not hold for some p, this implies that the Bergman space $A^p(D)$ differs from the space of those analytic functions f for which the corresponding weighted derivative (corresponding to $(1 - |z|^2)f'(z)$ in one dimension) belongs to $L^p(D, dV)$. These spaces are known as Besov spaces and have been studied in the case of bounded domains by several authors [1, 109, 110, 116]. We also raise the problem of understanding their theory in general unbounded domains of $\mathbb{C}^n$.

We consider the above three questions in this part of the thesis in the setting of tube domains over symmetric cones where some related work has been carried out in [13, 14, 22].

0.3.1 Bergman-type operators on tube domains over symmetric cones

Let V be a real vector space of dimension n, endowed with the structure of a simple Euclidean Jordan algebra. We consider an irreducible symmetric cone Ω inside $V = \mathbb{R}^n$ and denote by $T_\Omega = V + i\Omega$ the corresponding tube domain in the complexification of V. Here, V is endowed with an inner product $(\cdot|\cdot)$ for which the cone Ω is self-dual. These domains can be seen as multidimensional analogues of the upper half plane in $\mathbb{C}$. A typical example arises when Ω is the forward light-cone of $\mathbb{R}^n$, $n \geq 3$,

$$\Lambda_n = \left\{y \in \mathbb{R}^n \;:\; y_1^2 - y_2^2 - \ldots - y_n^2 > 0, \;\; y_1 > 0\right\}.$$

Other examples correspond to the cones $\mathrm{Sym}_+(r, \mathbb{R})$ of positive definite symmetric $r \times r$-matrices. We refer to the text [40] for a general description of symmetric cones. Following the notation in [40] we write r for the rank of Ω and $\Delta(x)$ for the associated determinant function. In the above examples, light-cones have rank 2 and determinant equal to the Lorentz form $\Delta(y) = y_1^2 - y_2^2 - \ldots - y_n^2$, while the cones $\mathrm{Sym}_+(r, \mathbb{R})$ have rank r and the determinant is the usual determinant of $r \times r$ matrices. We shall denote by $\mathcal{H}(T_\Omega)$ the space of holomorphic functions on T_Ω.

Given $1 \leq p, q < \infty$ and $\nu \in \mathbb{R}$, the mixed norm Lebesgue space $L^{p,q}_\nu(T_\Omega)$ is defined by the integrability condition

$$||f||_{L^{p,q}_\nu} := \left[\int_\Omega \left(\int_{\mathbb{R}^n} |f(x+iy)|^p dx\right)^{\frac{q}{p}} \Delta^{\nu - \frac{n}{r}}(y) dy\right]^{\frac{1}{q}} < \infty. \tag{0.3.3}$$

The mixed norm weighted Bergman space $A^{p,q}_\nu(T_\Omega)$ is then the closed subspace of $L^{p,q}_\nu(T_\Omega)$ consisting of holomorphic functions on the tube T_Ω. These spaces are nontrivial only when $\nu > \frac{n}{r} - 1$ (see [12]). When $p = q$ we shall simply write $A^{p,p}_\nu = A^p_\nu$. The usual Bergman space A^p then corresponds to the case $\nu = \frac{n}{r}$.

The weighted Bergman projection P_ν is the orthogonal projection from the Hilbert space $L^2_\nu(T_\Omega)$ onto its closed subspace $A^2_\nu(T_\Omega)$ and it is given by the integral formula

$$P_\nu f(z) = \int_{T_\Omega} B_\nu(z, w) f(w) \Delta^{\nu - \frac{n}{r}}(\Im w) dV(w) \tag{0.3.4}$$

where

$$B_\nu(z, w) = d_\nu \Delta^{-\nu - \frac{n}{r}}\left(\frac{z - \overline{w}}{i}\right) \tag{0.3.5}$$

is the weighted Bergman kernel, $d_\nu = \frac{2^{r\nu}}{(2\pi)^n} \frac{\Gamma_\Omega(\nu + \frac{n}{r})}{\Gamma_\Omega(\nu)}$ and dV is the Lebesgue measure on $\mathbb{C}^n$(see [12]).

The $L^{p,q}_\nu$-boundedness of the Bergman projection P_ν is still an open problem and has attracted a lot of attention in recent years (see [14], [10], [9], [13]). To date, it is only known that this projection extends as a bounded operator on $L^{p,q}_\nu$ for general symmetric cones for the range $1 \le p < \infty$ and $q'_{\nu,p} < q < q_{\nu,p}$, with $q_{\nu,p} = \min\{p, p'\}q_\nu$, $q_\nu = 1 + \frac{\nu}{\frac{n}{r}-1}$ and $\frac{1}{p} + \frac{1}{p'} = 1$ (see for example [13]) with slight improvements over this range in the case of light-cones (see [52]).

In this chapter, we consider the $L^{p,q}_\nu$-boundedness of a family of operators generalizing the Bergman projection . This family is given by the integral operators $T = T_{\alpha,\beta,\gamma}$ and $T^+ = T^+_{\alpha,\beta,\gamma}$ defined on $C^\infty_c(T_\Omega)$ by the formulas

$$Tf(z) = \Delta^\alpha(\Im z)\int_{T_\Omega} B_\gamma(z,w)f(w)\Delta^\beta(\Im w)dV(w),$$

and

$$T^+f(z) = \Delta^\alpha(\Im z)\int_{T_\Omega} |B_\gamma(z,w)|f(w)\Delta^\beta(\Im w)dV(w).$$

Remark 0.3.1. The boundedness of T^+ on $L^{p,q}_\nu(T_\Omega)$ implies the boundedness of T, although the boundedness of T is typically expected in a larger range than T^+.

The boundedness of this family of operators on $L^{p,q}_\nu(T_\Omega)$ has been considered in [14] for the case $P_\mu = T_{0,\mu-\frac{n}{r},\mu}$ and in [10] for $T_{0,\mu-\frac{n}{r},\mu+m}$. Both works deal with the case of the light cone. Here, we consider the problem of the boundedness of the operator T^+ for general symmetric cones and obtain optimal results for this operator. For this, we systematically make use of the methods of [14] and [10]. We mention that the case $p = q$ for general symmetric cones was implicit in [17]. Our results can be stated in the following way.

Theorem 0.3.2. *Suppose $\nu \in \mathbb{R}$ and $1 \le p, q < \infty$. Then the following conditions are equivalent:*

(a) The operator $T^+_{\alpha,\beta,\gamma}$ is bounded on $L^{p,q}_\nu(T_\Omega)$.

(b) The parameters satisfy $\gamma = \alpha + \beta + \frac{n}{r}$, $\alpha + \beta > -1$ and

$$\max\{-q\alpha+\frac{n}{r}-1, q(-\alpha+\frac{n}{r}-1)-\frac{n}{r}+1\} < \nu < \min\{q(\beta+1)+\frac{n}{r}-1, q(\beta+\frac{n}{r})-\frac{n}{r}+1\}.$$

Theorem 0.3.3. *The operator $T^+_{\alpha,\beta,\gamma}$ is bounded on $L^\infty(T_\Omega)$ if and only if $\alpha > \frac{n}{r} - 1$, $\beta > -1$ and $\gamma = \alpha + \beta + \frac{n}{r}$.*

As application, we characterize the dual space of Bergman spaces in some cases where the Bergman projection is not necessarily bounded.

0.3.2 Analytic Besov spaces in tubular domains over symmetric cones

We still denote by T_Ω the symmetric domain of tube type $T_\Omega = \mathbb{R}^n + i\Omega$ where Ω is an *irreducible symmetric cone* in $\mathbb{R}^n$.

A major open question in these domains concerns the L^p boundedness of the *Bergman projection* [9, 13, 14]. Let $A^p_\nu(T_\Omega)$ denote the subspace of holomorphic functions in $L^p_\nu = L^p(T_\Omega, \Delta(y)^{\nu - n/r} dx\, dy)$. These spaces are non trivial (i.e. $A^p_\nu \neq \{0\}$) only if $\nu > \frac{n}{r} - 1$ (see [38]). The usual (unweighted) Bergman spaces $A^p(T_\Omega)$ correspond to $\nu = \frac{n}{r}$. Let P_ν be the orthogonal projection mapping $L^2_\nu(T_\Omega)$ into $A^2_\nu(T_\Omega)$. Let us write

$$(1-x)_+ = \begin{cases} 1-x & \text{if} \quad 1-x>0 \\ 0 & \text{otherwise} \end{cases}$$

Conjecture 1. *Let $\nu > \frac{n}{r} - 1$. Then the Bergman projection P_ν admits a bounded extension to $L^p_\nu(T_\Omega)$ if and only if*

$$p'_\nu \; < \; p \; < \; p_\nu := \frac{\nu + \frac{2n}{r} - 1}{\frac{n}{r} - 1} - \frac{(1-\nu)_+}{\frac{n}{r} - 1}.$$

The necessity of the condition above was proved in [13]. The conjecture concerns the sufficiency. Note that the summand involving $(1-\nu)_+$ in the second term only occurs in the three dimensional forward light-cone ($n = 3$ and $r = 2$), where ν is allowed to take values below 1.

The problem of L^p-continuity of the Bergman projection has been studied in the papers [9, 11, 13, 14], and completely settled for large ν in the case of light cones in [13]. Let us write

$$\tilde{p}_\nu := \frac{\nu + \frac{2n}{r} - 1}{\frac{n}{r} - 1}.$$

Then the necessary condition $p < \tilde{p}_\nu$ is given by the fact that by duality, the Bergman kernel has to belong to the dual space $L^{p'}_\nu(T_\Omega)$. As far as sufficient conditions are concerned, we refer to [52, 53] for the best sufficient conditions that are known, up to now, in the case of light cones. In general, it is proved in [13, 14] that the Bergman projection P_ν is bounded in $L^p_\nu(T_\Omega)$ for

$$\bar{p}'_\nu < p < \bar{p}_\nu := \frac{\nu + \frac{2n}{r} - 2}{\frac{n}{r} - 1}.$$

Let $\Box = \Delta(\frac{1}{i}\frac{\partial}{\partial x})$ denote the differential operator of degree r defined by the equality:

$$\Box\, [e^{i(x|\xi)}] = \Delta(\xi) e^{i(x|\xi)}, \quad \xi \in \mathbb{R}^n. \tag{0.3.6}$$

In cones of rank 1 and 2 this corresponds to $-i\partial_x$ (when T_Ω is the upper-half-plane) and $-(\partial^2_{x_1} - \partial^2_{x_2} - \ldots - \partial^2_{x_n})/4$ (when T_Ω is the forward light cone), which justifies the name of "wave operator" given to Δ. We denote by $\Box_z$ the extension of the operator $\Box$ to $\mathbb{C}^n$ given by $\Box_z = \Delta(\frac{1}{i}\frac{\partial}{\partial x})$. When there is no ambiguity, we write $\Box$ instead of $\Box_z$.

In this chapter, instead of improving the above results, we will be concerned with equivalent formulations of Conjecture 1 and some consequences in the formulation of the theory of analytic Besov spaces in these settings. Our first result is the answer to the question of the equivalence between the boundedness of the Bergman projection and the validity of a Hardy inequality.

Theorem 0.3.4. *Let $\nu > \frac{n}{r} - 1$. Then, for $p \geq 2$, the Bergman projection P_ν admits a bounded extension to $L^p_\nu(T_\Omega)$ if and only if there exists a constant C such that, for all $F \in A^p_\nu$ we have*

$$\iint_{T_\Omega} |F(x+iy)|^p \, \Delta^{\nu-\frac{n}{r}}(y)\, dx\, dy \;\leq\; C \iint_{T_\Omega} \left|\Delta(y)\Box F(x+iy)\right|^p \Delta^{\nu-\frac{n}{r}}(y)\, dx\, dy. \quad (0.3.7)$$

Such an inequality is called a *Hardy Inequality* by reference to the one dimensional case.

We note that the reverse inequality always holds and that (0.3.7) is always valid when $1 \leq p \leq 2$, as it can be proved, for instance, from an explicit formula for F in terms of $\Box F$ involving the fundamental solution of the Box operator (see [22]). However, in this range (7.1.3) has no implications in terms of boundedness of Bergman projections. Hardy's inequalities have been also considered in [22]. In [14], for forward light cones, Hardy's inequalities were used as a key argument for proving the continuity of the Bergman projection.

The second equivalent formulation of Conjecture 1 concerns duality.

Theorem 0.3.5. *Let $\nu > \frac{n}{r} - 1$ and $1 < p < \infty$. Then P_ν admits a bounded extension to $L^p_\nu(T_\Omega)$ if and only if the natural mapping of $A^{p'}_\nu$ into $(A^p_\nu)^*$ is an isomorphism.*

Remark 0.3.6. If $p > \tilde{p}'_\nu$, then the inclusion $\Phi : A^{p'}_\nu \hookrightarrow (A^p_\nu)^*$ is injective, and hence boundedness of P_ν is actually equivalent to surjectivity of Φ. When $p \geq \tilde{p}_\nu$ these two properties fail, and $(A^p_\nu)^*$ is a space strictly larger than $A^{p'}_\nu$ which we do not know how to identify.

The two theorems above give two equivalent formulations of the boundedness of the Bergman projection for $p > 2$. They are proved in Section 3. When $1 \leq p < 2$ is such that the projection P_ν is not bounded, then we can still describe the dual space of A^p_ν in terms of equivalence classes of holomorphic functions, and more precisely in terms of Besov spaces. We define analytic Besov spaces $\mathbb{B}^p_\nu$, for $\nu \in \mathbb{R}$ and $1 \leq p < \infty$, by

$$\mathbb{B}^p_\nu := \{F \ : \ \Delta^m(\Im \cdot)\Box^m F \in L^p_\nu\}$$

for m large enough. The smallest possible value for m in the above definition is related to the validity of the Hardy inequality for some other weight, and one has to deal with equivalence classes modulo holomorphic functions that are annihilated by powers of the Box operator when m cannot be taken equal to 0. For the one dimensional case and bounded symmetric domains, we refer to [47, 115, 116]. Here, compared to the case of bounded symmetric domains, it is more difficult to deal with equivalence classes.

Let us mention the following special family of Besov spaces corresponding to the weight $\nu = -n/r$ in the above definition that is,

$$\mathbb{B}^p = \left\{ F \in \mathcal{H}(T_\Omega) \ : \ \Delta^m(\Im \cdot)\Box^m F \in L^p(d\lambda) \right\}.$$

Here $d\lambda = \Delta^{-\frac{2n}{r}}(y)dx\,dy$ denotes the invariant measure under conformal transformations of T_Ω. These are the analogue for T_Ω of the Besov spaces introduced by Arazy and Yan in bounded symmetric domains [1, 109, 110]. The space $\mathbb{B}^p$ is the right range of symbols of Hankel operators in the Schatten class $\mathcal{S}_p$ [24, 115]. For $p = \infty$, the Besov space is known as the Bloch space (see e.g. [7, 8]).

In Section 4, we study several properties of these spaces such as duality, integral representation, complex interpolation, and real analysis characterization in a point of view provided by [13]. We also discuss the problem of the minimum number of derivatives in the definition of Besov spaces.

0.3.3 Hankel operators on Bergman spaces of tube domains over symmetric cones

Let $b \in L^2(T_\Omega) = L^2(T_\Omega, dV)$. The small Hankel operator h_b with symbol b is defined as

$$h_b(f) = P(b\overline{f}) \tag{0.3.8}$$

for $f \in H^\infty(T_\Omega)$.

The aim of this chapter is to give criteria for Schatten class ($\mathcal{S}_p$) membership of Hankel operators on the Bergman space $A^2(T_\Omega)$. This problem has been considered in [2], [65] for the case of the unit disc of the complex plane, and in [116] and [115] for bounded symmetric domains. Some earlier works were done in [1], [35], [61], [80] and [90] in various domains including the upper half plane. It is shown in those cases that the small Hankel operator is in the Schatten class $\mathcal{S}_p$ if and only if its symbol belongs to the corresponding Besov space $\mathbb{B}^p$. Let us mention that the same problem for Hardy space of tube domains over symmetric cones was considered in [24] where it is stated that classical result extends to this case at least for $1 \leq p \leq 2$. Combining techniques of [24, 115], we prove that classical results (see [115] for example) extend to the tube domains over symmetric cones for the range $1 \leq p \leq \infty$. When the symbol is analytic and $1 \leq p \leq \infty$, we also obtain criteria in terms of the action of the operator on the reproducing kernel, here, "the reproducing kernel thesis". This last characterization appears in [98] for the same problem in the case of Hardy space of the unit disc.

Contents

Part I

Carleson-type measures in the unit ball of $\mathbb{C}^n$

Chapter 1

Preliminaries

In this chapter we introduce some basic properties of the unit ball. Results of this chapter will be used in the next chapter.

1.1 Basic properties of the unit ball

Let n be a positive integer and let

$$\mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C}$$

denote the n dimensional complex Euclidean space.

For $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ in $\mathbb{C}^n$, we write

$$\langle z, w \rangle = z_1 \overline{w_1} + \cdots + z_n \overline{w_n}$$

and

$$|z| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}.$$

The open unit ball in $\mathbb{C}^n$ is the set

$$\mathbb{B}^n = \{z \in \mathbb{C}^n : \quad |z| < 1\}.$$

The boundary of $\mathbb{B}^n$ will be denoted by $\mathbb{S}^n$ and is called the unit sphere in $\mathbb{C}^n$. Thus

$$\mathbb{S}^n = \{z \in \mathbb{C}^n : \quad |z| = 1\}.$$

Remark 1.1.1. In one dimension (when $n = 1$) one speaks of the unit disc that is the set

$$\mathbb{D} = \{z \in \mathbb{C} : \quad |z| < 1\}.$$

Its boundary is the unit circle $\mathbb{T}$ defined as

$$\mathbb{T} = \{z \in \mathbb{C} : \quad |z| = 1\}.$$

A function $f : \mathbb{B}^n \to \mathbb{C}$ is said to be holomorphic in $\mathbb{B}^n$ if

$$f(z) = \sum_m c(m) z^m, \qquad z \in \mathbb{B}^n. \tag{1.1.1}$$

Here the summation is over all multi-indexes

$$m = (m_1, \ldots, m_n),$$

where each m_k is a nonnegative integer and

$$z^m = z_1^{m_1} \cdots z_n^{m_n}.$$

The series in (1.1.1) is called the Taylor expansion of f at the origin; it converges absolutely and uniformly on each of the sets

$$r\mathbb{B}^n = \{z \in \mathbb{C}^n; \quad |z| \leq r\}, \qquad 0 < r < 1.$$

If we let

$$f_k(z) = \sum_{|m|=k} c(m) z^m$$

for each $k \geq 0$, where

$$|m| = m_1 + \cdots + m_n,$$

then the Taylor series of f can be rewritten as

$$f(z) = \sum_{k=0}^{\infty} f_k(z).$$

This is called the homogeneous expansion of f; each f_k is a homogeneous polynomial of degree k. Both the Taylor and the homogeneous expansion of f are uniquely determined by f.

For a multi-index $m = (m_1, \ldots, m_n)$ we will employ the notation

$$m! = m_1! \cdots m_n!.$$

In particular, we have the multinomial formula

$$(z_1 + \cdots + z_n)^N = \sum_{|m|=N} \frac{N!}{m!} z^m.$$

We will denoted by $\mathcal{H}(\mathbb{B}^n)$ the space of all holomorphic functions in $\mathbb{B}^n$.

1.1.1 The automorphism group

A mapping $F : \mathbb{B}^n \to \mathbb{C}^N$, where N is a positive integer, is given by N functions as follows:

$$F(z) = (f_1(z), \ldots, f_N(z)), \qquad z \in \mathbb{B}^n.$$

We say that F is a holomorphic mapping if each f_k is holomorphic in $\mathbb{B}^n$.

A mapping $F : \mathbb{B}^n \to \mathbb{B}^n$ is said to be bi-holomorphic if F is one-to-one and onto and F and its inverse, F^{-1}, are holomorphic.

The automorphism group of $\mathbb{B}^n$, denoted by $\mathrm{Aut}(\mathbb{B}^n)$, consists of all biholomorphic mappings of $\mathbb{B}^n$. If $\varphi \in \mathrm{Aut}(\mathbb{B}^n)$ with $\varphi(a) = 0$ ($a \neq 0$), then there is a linear fractional map φ_a on $\mathbb{B}^n$ and a unitary transformation of $\mathbb{C}^n$ such that $\varphi = U\varphi_a$. The fractional linear map φ_a is given by

$$\varphi_a(z) = \frac{a - P_a z - (1 - |a|^2)^{1/2} Q_a z}{1 - \langle z, a\rangle}\,, \qquad z \in \mathbb{B}^n, \tag{1.1.2}$$

where P_a is the orthogonal projection from $\mathbb{C}^n$ onto the one dimensional subspace $[a]$ generated by a, and $Q_a = I - P_a$. We clearly have

$$P_a = \langle \cdot, a\rangle a / \|a\|^2.$$

When $a = 0$, we simply take $\varphi_a(z) = -z$.

Lemma 1.1.2. *For each $a \in \mathbb{B}^n$ the mapping φ_a satisfies*

$$1 - \langle \varphi_a(z), \varphi_a(w)\rangle = \frac{(1 - \langle a, a\rangle)(1 - \langle z, w\rangle)}{(1 - \langle z, a\rangle)(1 - \langle a, w\rangle)} \tag{1.1.3}$$

for all z and w in the closed unit ball $\overline{\mathbb{B}^n} = \mathbb{B}^n \cup \mathbb{S}^n$.

Moreover, for each $a \in \mathbb{B}^n$,

$$\varphi_a \circ \varphi_a(z) = z\,, \qquad z \in \mathbb{B}^n.$$

In particular, each φ_a is an automorphism of $\mathbb{B}^n$ that interchanges the points 0 and a.

Observe that in (1.1.3), if we take $z = w$, we obtain the useful relation

$$1 - |\varphi_a(z)|^2 = \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - \langle z, a\rangle|^2}. \tag{1.1.4}$$

1.1.2 Lebesgue spaces

Most spaces considered in of the thesis will be defined in terms of L^p integrals of a function or its derivatives. The measures we use in these integrals for this part of the thesis are based on the volume measure on the unit ball or the surface measure on the unit sphere. We refer to [117] for the results in this section.

We let dV denote the volume measure on $\mathbb{B}^n$, normalized such that $V(\mathbb{B}^n) = 1$. The surface measure on $\mathbb{S}^n$ will be denoted by $d\sigma$. Once again, we normalize σ such that $\sigma(\mathbb{S}^n) = 1$. The next lemma gives an integration formula in polar coordinates.

Lemma 1.1.3. *The measures m and σ are related by*

$$\int_{\mathbb{B}^n} f(z)dV(z) = 2n \int_0^1 r^{2n-1}dr \int_{\mathbb{S}^n} f(r\xi)d\sigma(\xi).$$

dV and $d\sigma$ are invariant under unitary transformations. For $\alpha > -1$, we define the finite measure

$$dV_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha dV(z),$$

where c_α is a normalizing constant such that $V_\alpha(\mathbb{B}^n) = 1$. Using polar coordinates, we can show that

$$c_\alpha = \frac{\Gamma(n + \alpha + 1)}{n!\Gamma(\alpha + 1)}.$$

When $\alpha \leq -1$, we simply write

$$dV_\alpha(z) = (1 - |z|^2)^\alpha dm(z).$$

All the measures dV_α, $\alpha \in \mathbb{R}$, are also unitary invariant (or rotation invariant), that is,

$$\int_{\mathbb{B}^n} f(Uz)dV_\alpha(z) = \int_{\mathbb{B}^n} f(z)dV_\alpha(z)$$

for all $f \in L^1(\mathbb{B}^n, dV_\alpha)$ and all unitary transformations U of $\mathbb{C}^n$. As a consequence, we obtain that

$$\int_{\mathbb{S}^n} \xi^m\overline{\xi^l}d\sigma(\xi) = 0, \qquad \int_{\mathbb{B}^n} z^m\overline{z^l}dV_\alpha(z) = 0,$$

if m and l are multi-indexes of nonnegative integers with $m \neq l$. When $m = l$, we have the following formulas.

Lemma 1.1.4. *Suppose $m = (m_1, \ldots, m_n)$ is a multi-index of nonnegative integers and $\alpha > -1$. Then*

$$\int_{\mathbb{S}^n} |\xi^m|^2 d\sigma(\xi) = \frac{(n-1)!m!}{(n-1+|m|)!},$$

and

$$\int_{\mathbb{B}^n} |z^m|^2 dV_\alpha(z) = \frac{m!\Gamma(n+\alpha+1)}{\Gamma(n+|m|+\alpha+1)}.$$

As a consequence of the above lemma and Stirling's formula, one obtain the following asymptotic estimates for certain important integrals on the ball and the sphere.

Theorem 1.1.5. *Suppose s is real and $t > -1$. Then the integrals*

$$I_s(z) = \int_{\mathbb{S}^n} \frac{d\sigma(\xi)}{|1-\langle z,\xi\rangle|^{n+s}}, \qquad z \in \mathbb{B}^n,$$

and

$$J_{s,t}(z) = \int_{\mathbb{B}^n} \frac{(1-|w|^2)^t dV(w)}{|1-\langle z,w\rangle|^{n+1+t+s}}, \qquad z \in \mathbb{B}^n,$$

have the following asymptotic properties.

(1) If $s < 0$, then I_s and $J_{s,t}$ are both bounded in $\mathbb{B}^n$.

(2) If $s = 0$, then

$$I_s(z) \sim J_{s,t}(z) \sim \log \frac{1}{1-|z|^2}$$

as $|z| \to 1^-$.

(3) If $s > 0$, then

$$I_s(z) \sim J_{s,t}(z) \sim (1-|z|^2)^{-s}.$$

as $|z| \to 1^-$.

The notation $A \sim B$ means that one can find a positive constant M such that

$$\frac{1}{M}B \leq A \leq MB.$$

1.2 Various derivatives of a holomorphic function

In this section we introduce some notions of differentiation on $\mathbb{B}^n$ that we will need in the next chapter. The most basic one is the standard partial differentiation, that is $\frac{\partial f}{\partial z}$. We also give a well-known integration formula. We first introduce the very important notion of the radial derivative in the unit ball.

The radial derivative Rf of a holomorphic function f is given by

$$Rf(z) = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z).$$

For a holomorphic function f in $\mathbb{B}^n$ we write

$$\nabla f(z) = \left(\frac{\partial f}{\partial z_1}(z), \cdots, \frac{\partial f}{\partial z_k}(z)\right) \tag{1.2.1}$$

and call $|\nabla f(z)|$ the (holomorphic) gradient of f at z. We also define

$$\tilde{\nabla} f(z) = \nabla(f \circ \varphi_z)(0) \tag{1.2.2}$$

where φ_z is the biholomorphic mapping of $\mathbb{B}^n$ that interchanges 0 and z, and call $|\tilde{\nabla} f(z)|$ the invariant gradient of f at z.

Lemma 1.2.1. *If f is holomorphic in $\mathbb{B}^n$, then*

$$|\tilde{\nabla} f(z)|^2 = (1-|z|^2)(|\nabla f(z)|^2 - |Rf(z)|^2)$$

for all $z \in \mathbb{B}^n$.

The proof of the above lemma which uses the notion of the invariant Laplacian can be found in [117].

Lemma 1.2.2. *If f is holomorphic in $\mathbb{B}^n$, then*

$$(1-|z|^2)|Rf(z)| \leq (1-|z|^2)|\nabla f(z)| \leq |\tilde{\nabla} f(z)|$$

for all $z \in \mathbb{B}^n$.

Proof. By the Cauchy-Schwarz inequality for $\mathbb{C}^n$,

$$|Rf(z)| \leq |z||\nabla f(z)| \leq |\nabla f(z)|.$$

This gives the first inequality. Now, using the inequality $|Rf(z)| \leq |z||\nabla f(z)|$, we obtain from Lemma 1.2.1:

$$|\tilde{\nabla} f(z)|^2 \geq (1-|z|^2)(|\nabla f(z)|^2 - |z|^2|\nabla f(z)|^2) = (1-|z|^2)^2|\nabla f(z)|^2.$$

The proof is complete. □

The following lemma can be proved using integration by parts (see [92]) .

Lemma 1.2.3. *Let f, g be holomorphic polynomials on $\mathbb{B}^n$. Then the following identity holds*

$$\begin{aligned} \int_{\mathbb{S}^n} f(\xi)\overline{g}(\xi)d\sigma(\xi) &= C_1 \int_{\mathbb{B}^n} f(z)\overline{g(z)}dV(z) + C_2 \int_{\mathbb{B}^n} Rf(z)\overline{g(z)}(1-|z|^2)dV(z) + \\ & \quad C_3 \int_{\mathbb{B}^n} f(z)\overline{Rg(z)}(1-|z|^2)dV(z) \end{aligned}$$

C_1, C_2 and C_3 being constants independent of f and g.

1.3 The Bergman Metric

The function

$$K(z,w)=\frac{1}{(1-\langle z,w\rangle)^{n+1}}$$

is called the Bergman kernel of $\mathbb{B}^n$. The Bergman matrix is the $n\times n$ complex matrix $B(z)=(b_{ij}(z))$ with entries

$$b_{ij}(z)=\frac{\partial^2}{\partial\overline{z}_i\partial z_j}\log K(z,z).$$

The Bergman matrix is positive and invertible. Moreover, it is invariant under automorphisms of $\mathbb{B}^n$, that is,

$$B(z)=(\varphi'(z))^*B(\varphi(z))\varphi'(z)$$

for all $z\in\mathbb{B}^n$ and $\varphi\in Aut(\mathbb{B}^n)$.

For a smooth curve $\gamma:[0,1]\to\mathbb{B}^n$ we define

$$\begin{aligned}\ell(\gamma) &= \int_0^1\left(\sum_{i,j=1}^n b_{ij}(\gamma(t))\dot{\gamma}_i(t)\overline{\dot{\gamma}_j(t)}\right)^{1/2} dt\\ &= \int_0^1\langle B(\gamma(t))\gamma'(t),\gamma'(t)\rangle^{1/2}dt.\end{aligned}$$

This definition generalizes to the case of piecewise smooth curves. Thus we can define metric $\beta:\mathbb{B}^n\times\mathbb{B}^n\to[0,\infty)$ as follows: for any two points z and w in $\mathbb{B}^n$, let $\beta(z,w)$ be the infimum of the set consisting of all $\ell(\gamma)$, where γ is a piecewise smooth curve in $\mathbb{B}^n$ from z to w. That β is a metric follows easily from the positivity of $B(z)$. β is called the Bergman metric on $\mathbb{B}^n$. The following proposition is a consequence of the invariance of the Bergman matrix under automorphism.

Proposition 1.3.1. *The Bergman metric is invariant under automorphisms, that is,*

$$\beta(\varphi(z),\varphi(w))=\beta(z,w)$$

for all $z,\ w\in\mathbb{B}^n$ and $\varphi\in Aut(\mathbb{B}^n)$.

Using invariance again and other easy properties of the Bergman matrix, one proves the following.

Proposition 1.3.2. *If z and w are points in $\mathbb{B}^n$, then*

$$\beta(z,w)=\frac{1}{2}\log\frac{1+\varphi_z(w)}{1-\varphi_z(w)},$$

where φ_z is the involutive automorphism of $\mathbb{B}^n$ that interchanges 0 and z.

For $a \in \mathbb{B}^n$ and $r > 0$ we let $D(a, r)$ denote the Bergman metric ball centered at a with radius r. Thus

$$D(a, r) = \{z \in \mathbb{B}^n : \beta(a, z) < r\}.$$

For any $\xi \in \mathbb{S}^n$ and $\delta > 0$, let

$$Q_\delta(\xi) = \{z \in \mathbb{B}^n : |1 - \langle z, \xi \rangle| < \delta\}.$$

These are the higher dimension analogues of Carleson regions. We have the following asymptotic estimates of the volumes $V_\alpha(D(a, r)) = \int_{D(a,r)} dV_\alpha(z)$ and $V_\alpha(Q_\delta(\xi)) = \int_{Q_\delta(\xi)} dV_\alpha(z)$.

Lemma 1.3.3. *For any real $\alpha \in \mathbb{R}$ and $r > 0$ there exist constants $C_{\alpha,r} > 0$ and $c_{\alpha,r} > 0$ such that*

$$c_{\alpha,r}(1 - |a|^2)^{n+1+\alpha} \leq V_\alpha(D(a, r)) \leq C_{\alpha,r}(1 - |a|^2)^{n+1+\alpha}$$

for all $a \in \mathbb{B}^n$.

Lemma 1.3.4. *For any $\alpha > -1$ there exist constants $C_\alpha > 0$ and $c_\alpha > 0$ such that*

$$c_\alpha \delta^{n+1+\alpha} \leq V_\alpha(Q_\delta(\xi)) \leq C_\alpha \delta^{n+1+\alpha}$$

for all $\xi \in \mathbb{S}^n$ and $0 \leq \delta \leq 1$.

Remark 1.3.5. We refer to [117] for the proof of results of this section.

Chapter 2

On some equivalent definitions of ρ- Carleson measures on the unit ball

We give in this chapter some equivalent definitions of the so called ρ-Carleson measures when $\rho(t) = (\log(4/t))^p(\log\log(e^4/t))^q$, $0 \leq p, q < \infty$. As applications, we characterize the pointwise multipliers on $LMOA(\mathbb{S}^n)$ and from this space to $BMOA(\mathbb{S}^n)$. Boundedness of the Cesàro type integral operators on $LMOA(\mathbb{S}^n)$ and from $LMOA(\mathbb{S}^n)$ to $BMOA(\mathbb{S}^n)$ is considered as well. The case $\rho(t) = (\log(4/t))^p$ was considered in [111] which also inspired this work.

2.1 Holomorphic function spaces and Carleson measures in the unit ball

2.1.1 Some holomorphic function spaces in the unit ball of $\mathbb{C}^n$

We define here various holomorphic function spaces appearing in this chapter. We refer to the book [117] for the proof of different assertions stated below.

Recall that for $\alpha > -1$ the weighted Lebesgue measure dV_α is defined by

$$dV_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha dV(z), \tag{2.1.1}$$

where

$$c_\alpha = \frac{\Gamma(n + \alpha + 1)}{n!\Gamma(\alpha + 1)} \tag{2.1.2}$$

is a normalizing constant so that $V_\alpha(\mathbb{B}^n) = 1$.

Definition 2.1.1. For $\alpha > -1$ and $0 < p < \infty$, the weighted Bergman space $A^p_\alpha(\mathbb{B}^n)$ consists of the holomorphic functions f in $L^p(\mathbb{B}^n, dV_\alpha)$, that is

$$A^p_\alpha(\mathbb{B}^n) = L^p(\mathbb{B}^n, dV_\alpha) \bigcap H(\mathbb{B}^n). \tag{2.1.3}$$

We use the notation

$$||f||^p_{p,\alpha} := \int_{\mathbb{B}^n} |f(z)|^p dV_\alpha(z) \tag{2.1.4}$$

for $f \in L^p(\mathbb{B}^n, dV_\alpha)$.

Definition 2.1.2. For $0 < p < \infty$ the Hardy space $\mathcal{H}^p(\mathbb{B}^n)$ is the space of all $f \in H(\mathbb{B}^n)$ such that

$$||f||^p_p := \sup_{0<r<1} \int_{\mathbb{S}^n} |f(r\xi)|^p d\sigma(\xi) < \infty. \tag{2.1.5}$$

The space of all bounded holomorphic functions in $\mathbb{B}^n$ will be denoted $\mathcal{H}^\infty(\mathbb{B}^n)$.

For any $\xi \in \mathbb{S}^n$ and $\delta > 0$, let

$$B_\delta(\xi) = \{w \in \mathbb{S}^n : |1 - \langle w, \xi\rangle| < \delta\},$$

and

$$Q_\delta(\xi) = \{z \in \mathbb{B}^n : |1 - \langle z, \xi\rangle| < \delta\}.$$

These are the higher dimension analogues of Carleson regions. For $f \in \mathcal{H}^1(\mathbb{B}^n)$, we still denote $f(\xi)$, for $\xi \in \mathbb{S}^n$, the admissible limit at the boundary, which exists a.e (see e.g. [92]).

We recall that the space of functions of bounded mean oscillation in $\mathbb{B}^n$ $BMOA(\mathbb{S}^n)$ is the space of all $f \in \mathcal{H}^1(\mathbb{B}^n)$ such that

$$\sup_{\substack{B=B_\delta(\xi),\\ \delta\in(0,1),\xi\in\mathbb{S}^n}} \frac{1}{\sigma(B)} \int_B |f - f_B| d\sigma \leq C.$$

Here and anywhere else, f_B denotes the mean-value of f on B.

The space $BMOA$ is Banach space when equipped with the norm

$$||f||_{BMOA} = |f(0)| + \sup_{\substack{B=B_\delta(\xi),\\ \delta\in(0,1),\xi\in\mathbb{S}^n}} \frac{1}{\sigma(B)} \int_B |f - f_B| d\sigma.$$

We now define the space of functions of logarithmic mean oscillation $LMOA$.

Definition 2.1.3. A function f belongs to $LMOA$ if $f \in \mathcal{H}^1(\mathbb{B}^n)$ and there exists a constant $C > 0$ so that

$$\sup_{\substack{B=B_\delta(\xi),\\ \delta\in]0,1[,\xi\in\mathbb{S}^n}} \frac{\log\frac{4}{\delta}}{\sigma(B)} \int_B |f - f_B| d\sigma \leq C.$$

The space $LMOA$ is Banach space when equipped with the norm

$$||f||_{LMOA} = |f(0)| + \sup_{\substack{B=B_\delta(\xi),\\ \delta\in]0,1[,\xi\in\mathbb{S}^n}} \frac{\log\frac{4}{\delta}}{\sigma(B)}\int_B |f-f_B|d\sigma.$$

The spaces $BMOA$ and $LMOA$ belong both to a more general class of holomorphic functions.

Definition 2.1.4. Let ρ be a positive non-increasing function defined on $(0,1)$. The space of functions of ρ-bounded mean oscillation $BMOA_\rho$ is the space of all $f\in\mathcal{H}^1(\mathbb{B}^n)$ for which there exists a constant $C>0$ so that

$$\sup_{\substack{B=B_\delta(\xi),\\ \delta\in(0,1),\xi\in\mathbb{S}^n}} \frac{\rho(\delta)}{\sigma(B)}\int_B |f-f_B|d\sigma \le C.$$

Here and anywhere else, f_B denotes the mean-value of f on B.

The space $BMOA_\rho$ is Banach space when equipped with the norm

$$||f||_{BMOA_\rho} = |f(0)| + \sup_{\substack{B=B_\delta(\xi),\\ \delta\in]0,1[,\xi\in\mathbb{S}^n}} \frac{\rho(\delta)}{\sigma(B)}\int_B |f-f_B|d\sigma$$

(see [99, 117]). When ρ is a constant function, the above space is the usual space of functions of bounded mean oscillation $BMOA$ and for $\rho(t)=\log(\frac{1}{t})$ this corresponds to the space of functions of logarithmic mean oscillation $LMOA$

Let us recall the following definition of the Bloch space of the unit ball of $\mathbb{C}^n$.

Definition 2.1.5. The Bloch space $\mathcal{B}$ consists of all $f\in H(\mathbb{B}^n)$ such that

$$||f||_{\mathcal{B}} = |f(0)| + \sup_{z\in B_n} |Rf(z)|(1-|z|^2) < \infty \tag{2.1.6}$$

where Rf is the radial derivative of f defined in the previous chapter.

We now introduce the following generalized α-logarithmic-type Bloch spaces.

Definition 2.1.6. For $0\le p,q<\infty$ and $\alpha>0$. Let $\mathcal{B}_\alpha^{p,q}$ denote the space of holomorphic functions f such that

$$\sup_{z\in\mathbb{B}^n}(1-|z|^2)^\alpha|Rf(z)|(\log\frac{4}{1-|z|^2})^p(\log\log\frac{e^4}{1-|z|^2})^q<\infty.$$

These can be seen as special case of the so called μ-Bloch spaces (see for example [62]) and one has that $\mathcal{B}_\alpha^{p,q}$ are Banach spaces with the norm

$$||f||_{\mathcal{B}_\alpha^{p,q}} \quad = \quad |f(0)| + \sup_{z\in\mathbb{B}^n}(1-|z|^2)^\alpha|Rf(z)|(\log\frac{4}{1-|z|^2})^p(\log\log\frac{e^4}{1-|z|^2})^q<\infty.$$

The usual Bloch space $\mathcal{B}$ then corresponds to the case $\alpha = 1$ and $p = q = 0$ while $\mathcal{B}_\alpha^{0,0} = \mathcal{B}_\alpha$ are the so called α- Bloch spaces (see [117]) and $\mathcal{B}_1^{1,1} = L\mathcal{B}$ is the so-called logarithmic Bloch space. Moreover, $BMOA$ continuously embeds in $\mathcal{B}$ and $LMOA$ embeds continuously in $L\mathcal{B}$ (see [117]).

2.1.2 Carleson measures on the unit ball of $\mathbb{C}^n$

We recall here the definition of Carleson measures and their equivalent characterizations in the unit ball. We also introduce Carleson measures with weight.

Definition 2.1.7. Let μ denote a positive Borel measure on $\mathbb{B}^n$. Then for $0 < s < \infty$, the measure μ is called a s-Carleson measure, if there is a finite constant $C > 0$ such that for any $\xi \in \mathbb{S}^n$ and any $0 < \delta < 1$,

$$\mu(Q_\delta(\xi)) \leq C(\sigma(B_\delta(\xi)))^s. \tag{2.1.7}$$

When $s = 1$, μ is just called Carleson measure. The infinimum of all these constants C will be denoted by $||\mu||_s$. We will also use $||\mu||$ to denote $||\mu||_1$. The following theorem is the higher dimension version of the theorem of L.Carleson [27] and its reproducing kernel formulation.

Theorem 2.1.8. *For a positive Borel measure μ on $\mathbb{B}^n$, and $0 < p < \infty$, the following are equivalent*

i) The measure μ is a Carleson measure

ii) There is a constant $C_1 > 0$ such that for all $f \in \mathcal{H}^p(\mathbb{B}^n)$,

$$\int_{\mathbb{B}^n} |f(z)|^p d\mu(z) \leq C_1 ||f||_p^p.$$

iii) There is a constant $C_2 > 0$ such that for all $a \in \mathbb{B}^n$,

$$\int_{\mathbb{B}^n} \frac{(1-|a|^2)^n}{|1-\langle a, w\rangle|^{2n}} d\mu(w) < C_2.$$

We say two positive constant K_1 and K_2 are comparable, denoted by $K_1 \approx K_2$, if there is an absolute positive constant M such that

$$M^{-1} \leq \frac{K_1}{K_2} \leq M.$$

We note that the constants C_1, C_2 in Theorem 2.1.8 are both comparable to $||\mu||$. Assertion $ii)$ in Theorem 2.1.8 is the usual definition of Carleson measures (for Hardy spaces). The characterization of these measures in the unit disc is due to Carleson [27] and its extension to the unit ball is due to Hörmander [60]. The proof of the above theorem can be found in [117].

The characterization of Carleson measures for Bergman spaces in the unit ball of $\mathbb{C}^n$ is due to Cima and Wogen [32]. We have the following theorem in [117] and [113].

Theorem 2.1.9. *For a positive Borel measure μ on $\mathbb{B}^n$, $s > 1$ and $0 < p < \infty$, the following are equivalent*

i) The measure μ is a s-Carleson measure

ii) There is a constant $K_1 > 0$ such that, for all $f \in A^p_{ns-(n+1)}$,

$$\int_{\mathbb{B}^n} |f(z)|^p d\mu(z) \leq K_1 ||f||^p_{p,ns-(n+1)}.$$

iii) There is a constant $K_2 > 0$ such that, for all $a \in \mathbb{B}^n$,

$$\int_{\mathbb{B}^n} \frac{(1-|a|^2)^{ns}}{|1-\langle a,w\rangle|^{2ns}} d\mu(w) < K_2.$$

Here both K_1 and K_2 are comparable to $||\mu||_s$. We consider here generalized Carleson type measures with additional logarithmic terms.

Definition 2.1.10. Let μ be a positive Borel measure on $\mathbb{B}^n$ and $0 < s < \infty$. For ρ a positive function defined on $(0,1)$, we say that μ is a (ρ, s)- Carleson measure if there is a constant $C > 0$ such that for any $\xi \in \mathbb{S}^n$ and $0 < \delta < 1$,

$$\mu(Q_\delta(\xi)) \leq C \frac{(\sigma(B_\delta(\xi)))^s}{\rho(\delta)}. \tag{2.1.8}$$

When $s = 1$, μ is called a ρ-Carleson measure. There is a close relation between ρ-Carleson measures and $BMOA_\rho$ space. The following is proved in [99].

Proposition 2.1.11. *A holomorphic function f belongs to $BMOA_\rho$ if and only if the measure $(1-|z|^2)|\nabla f(z)|^2 dV(z)$ is a ρ^2-Carleson measure.*

We are interested in this chapter in the particular case

$$\rho(t) = \rho_{p,q}(t) = (\log(4/t))^p (\log\log(e^4/t))^q$$

with $0 \leq p, q < \infty$. We remark that the case $\rho(t) = (\log(4/t))^p$ has been studied in [111] for the unit disc of the complex plane $\mathbb{C}$. The corresponding measures in the latter are called p-logarithmic s-Carleson measures. When $s = 1$ we call them p-logarithmic Carleson measures and when $p = 2$ and $s = 1$ we call them logarithmic Carleson measures, using the vocabulary of [111].

Let φ_z be the involutive automorphism of $\mathbb{B}^n$ that interchanges 0 and z. We recall that the Bergman metric of $\mathbb{B}^n$ is given by

$$\beta(z, w) := \frac{1}{2} \log \frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|},$$

for all $z, w \in \mathbb{B}^n$. For any $R > 0$ and any $a \in \mathbb{B}^n$, we write

$$D(a, R) = \{z \in \mathbb{B}^n : \beta(z, a) < R\}$$

for the Bergman ball centered at a with radius R. We have the following characterization of elements of $\mathcal{B}^{p,q}_\alpha$ in terms of weighted Carleson measures.

Lemma 2.1.12. *Let $0 \leq p, q < \infty$ and $\alpha > 0$. A function $f \in H(\mathbb{B}^n)$ is in $\mathcal{B}^{p,q}_\alpha$ if and only if $(1 - |z|^2)^{n(s-1)+2\alpha-1}|Rf(z)|^2 dV(z)$ is a $(\rho_{p,q}, s)$-Carleson measure for any $s > 1$, where $\rho_{p,q}(t) = (\log \frac{4}{t})^{2p}(\log\log \frac{e^4}{t})^{2q}$.*

Proof. We first suppose that f belongs to $\mathcal{B}^{p,q}_\alpha$ and show that there exists a constant $C > 0$ such that for any $\xi \in \mathbb{S}^n$, $0 < \delta < 1$ and any $s > 1$, the following inequality holds

$$I_f(\delta) \leq C\sigma(B_\delta(\xi))^s,$$

where

$$I_f(\delta) = (\log \frac{4}{\delta})^{2p}(\log\log \frac{e^4}{\delta})^{2q} \int_{Q_\delta(\xi)} |Rf(z)|^2(1 - |z|^2)^{n(s-1)+2\alpha-1} dV(z).$$

Let $h(x) = (\log \frac{4}{x})^{2p}(\log\log \frac{e^4}{x})^{2q}$. Then h is decreasing on $(0, 1)$ and moreover, for any $z \in Q_\delta(\xi)$, $1 - |z|^2 < |1 - \langle \xi, z\rangle| < \delta$. It follows using the definition of $\mathcal{B}^{p,q}_\alpha$ that there exists a constant $C > 0$ such that for all $f \in \mathcal{B}^{p,q}_\alpha$,

$$\begin{aligned} I_f(\delta) &\leq C \int_{Q_\delta(\xi)} \frac{h(\delta)}{h(1 - |z|^2)}(1 - |z|^2)^{n(s-1)-1} dV(z) \\ &\leq C \int_{Q_\delta(\xi)} (1 - |z|^2)^{n(s-1)-1} dV(z) \\ &\leq C\sigma(B_\delta(\xi))^s. \end{aligned}$$

This shows the necessary part.

Conversely, let us suppose that the analytic function f has the property that there exists $C > 0$ such that for any $\xi \in \mathbb{S}^n$, $0 < \delta < 1$ and any $s > 1$,

$$I_f(\delta) \leq C(\sigma(B_\delta(\xi)))^s$$

and show that in this case f belongs to $\mathcal{B}_\alpha^{p,q}$. We recall that for $a \in \mathbb{B}^n - \{0\}$ and $R > 0$; letting $\delta = 1 - |a|$, there exists $\lambda \in (0,1)$ (depending on R but not on δ)such that $D(a,R) \subset Q_\delta(\xi)$ with $a = (1-\lambda\delta)\xi$ (see [117, Lemma 5.23]). Now, using the mean value property, we obtain that for any $a \in \mathbb{B}^n$,

$$|Rf(a)|^2 \leq \frac{C}{(1-|a|^2)^{ns+2\alpha}} \int_{D(a,R)} |Rf(z)|^2(1-|z|^2)^{n(s-1)+2\alpha-1} dV(z).$$

It follows from the above inclusion and the hypotheses on the measure $|Rf(z)|^2(1-|z|^2)^{n(s-1)+2\alpha-1}dV(z)$ that

$$(1-|a|^2)^{2\alpha}|Rf(a)|^2(\log\frac{4}{1-|a|^2})^{2p}(\log\log\frac{e^4}{1-|a|^2})^{2q} \leq \frac{C}{\delta^{ns}} I_f(\delta) \leq C < \infty.$$

The proof is complete. $\square$

2.2 The case of $\rho_{p,q}$- Carleson measures

2.2.1 Some useful results

We give in this subsection some useful results for the characterizations of $\rho_{p,q}$- Carleson measures in the unit ball.

Lemma 2.2.1. *Let $1 < N < \infty$ and $0 < \alpha < \infty$. The following assertions hold.*

i) For any $0 \leq p < \infty$, there exists a positive constant C_1 not depending on N so that

$$I_{N,\alpha,p} = \int_1^N \frac{e^{-\alpha t}dt}{(N-t+2)^p} \leq \frac{C_1}{(N+2)^p}.$$

ii) If ϵ_1 and ϵ_2 are real with $\log(2+\epsilon_1)+\epsilon_2 > 1$, then for any $0 \leq p < \infty$, there exists a positive constant C_2 not depending on N so that

$$J_{N,\alpha,p} = \int_1^N \frac{e^{-\alpha t}dt}{(\log(N-t+2+\epsilon_1)+\epsilon_2)^p} \leq \frac{C_2}{(\log(N+2+\epsilon_1)+\epsilon_2)^p}.$$

Proof. i). A simple change of variables gives the following equalities

$$I_{N,\alpha,p} = \int_2^{N+1} \frac{e^{-\alpha(N+2-x)}dx}{x^p} = e^{-\alpha(N+2)} \int_2^{N+1} x^{-p}e^{\alpha x}dx.$$

Thus $i)$ can be written as

$$\int_2^{N+1} x^{-p}e^{\alpha x}dx \leq C_1(N+2)^{-p}e^{\alpha(N+2)}. \tag{2.2.1}$$

Let $f(x) = x^{-p}e^{\alpha x}$. Then $f'(x) = x^{-p-1}e^{\alpha x}(\alpha x - p) > 0$ if $x > \frac{p}{\alpha}$. Since $f(x)$ is obviously continuous on $[2, \infty)$ and increasing as $x > \frac{p}{\alpha}$, there is a positive constant K such that for any $x \in [2, N+1]$,

$$f(x) \leq Kf(N+1) = K(N+1)^{-p}e^{\alpha(N+1)}.$$

Integrating by parts gives

$$\begin{aligned}
\int_2^{N+1} x^{-p}e^{\alpha x}dx &= \frac{1}{\alpha}x^{-p}e^{\alpha x}\Big|_2^{N+1} + \frac{p}{\alpha}\int_2^{N+1} x^{-p-1}e^{\alpha x}dx \\
&\leq \frac{K}{\alpha}(N+1)^{-p}e^{\alpha(N+1)} + \frac{Kp}{\alpha}(N+1)^{-p-1}e^{\alpha(N+1)}\int_2^{N+1} dx \\
&\leq \frac{K(1+p)}{\alpha}(N+1)^{-p}e^{\alpha(N+1)} \\
&\leq \frac{K'(1+p)}{\alpha}(N+2)^{-p}e^{\alpha(N+2)},
\end{aligned}$$

where K' is another positive constant, independent of N. Thus (2.2.1) is true, hence (i) is true.

$ii)$. The proof is similar to the proof of $i)$. Let $x = N+2+\epsilon_1-t$. Then $t = N+2+\epsilon_1-x$, and $dt = -dx$. Thus

$$J_{N,\alpha,p} = \int_{2+\epsilon_1}^{N+1+\epsilon_1} \frac{e^{-\alpha(N+2+\epsilon_1-x)}dx}{(\epsilon_2+\log x)^p} = e^{-\alpha(N+2+\epsilon_1)}\int_{2+\epsilon_1}^{N+1+\epsilon_1}(\epsilon_2+\log x)^{-p}e^{\alpha x}dx.$$

Thus $ii)$ can be written as

$$\int_{2+\epsilon_1}^{N+1+\epsilon_1}(\epsilon_2+\log x)^{-p}e^{\alpha x}dx \leq C_2[\epsilon_2+\log(N+2+\epsilon_1)]^{-p}e^{\alpha(N+2+\epsilon_1)}. \tag{2.2.2}$$

Let $g(x) = (\epsilon_2+\log x)^{-p}e^{\alpha x}$. Then

$$g'(x) = (\epsilon_2+\log x)^{-p-1}e^{\alpha x}\left[\alpha(\epsilon_2+\log x) - \frac{p}{x}\right].$$

Since

$$\lim_{x\to\infty}\left[\alpha(\epsilon_2+\log x) - \frac{p}{x}\right] = \infty,$$

we know that there exists a positive constant M such that $g'(x) > 0$ for all $x > M$. Thus g is continuous on $[2+\epsilon_1, \infty)$ and increasing whenever $x > M$. Therefore there is a positive constant K_1 such that for any $x \in [2+\epsilon_1, N+1+\epsilon_1]$,

$$g(x) \leq K_1 g(N+1+\epsilon_1) = K_1[\epsilon_2+\log(N+1+\epsilon_1)]^{-p}e^{\alpha(N+1+\epsilon_1)}.$$

Integrating by parts gives

$$\begin{aligned}
\int_{2+\epsilon_1}^{N+1+\epsilon_1} (\epsilon_2+\log x)^{-p} e^{\alpha x} dx &= \frac{1}{\alpha}(\epsilon_2+\log x)^{-p} e^{\alpha x}\Big|_{2+\epsilon_1}^{N+1+\epsilon_1} + \frac{p}{\alpha}\int_{2+\epsilon_1}^{N+1+\epsilon_1} (\epsilon_2+\log x)^{-p-1} e^{\alpha x} x^{-1} dx \\
&\leq \frac{K_1}{\alpha}[\epsilon_2+\log(N+1+\epsilon_1)]^{-p} e^{\alpha(N+1+\epsilon_1)} \\
&\quad + \frac{K_1 p}{\alpha}[\epsilon_2+\log(N+1+\epsilon_1)]^{-p-1} e^{\alpha(N+1+\epsilon_1)} \int_{2+\epsilon_1}^{N+1+\epsilon_1} x^{-1} dx \\
&\leq \frac{K_1(1+p)}{\alpha}[\epsilon_2+\log(N+1+\epsilon_1)]^{-p} e^{\alpha(N+1+\epsilon_1)} \\
&\leq \frac{K_2(1+p)}{\alpha}[\epsilon_2+\log(N+2+\epsilon_1)]^{-p} e^{\alpha(N+2+\epsilon_1)},
\end{aligned}$$

where K_2 is another positive constant, independent of N. Thus (2.2.2) is true, hence *ii*) is true. The proof is complete.

□

Let

$$K_a(z) = \frac{(1-|a|^2)^n}{|1-\langle a,z\rangle|^{2n}}.$$

We have the following general characterizing of $(\rho_{p,q}, s)$-Carleson measures in the unit ball.

Theorem 2.2.2. *Let $0 \leq p, q < \infty$ and $0 < s < \infty$. Let μ be a positive Borel measure on $\mathbb{B}^n$. Then μ is a $(\rho_{p,q}, s)$-Carleson measure with $\rho_{p,q}(t) = (\log(4/t))^p(\log\log(e^4/t))^q$, if and only if*

$$\sup_{a\in\mathbb{B}^n}(\log\frac{4}{1-|a|})^p(\log\log\frac{e^4}{1-|a|})^q\int_{\mathbb{B}^n} K_a^s(z)d\mu(z) \leq C < \infty. \tag{2.2.3}$$

Proof. We first suppose that μ is a $(\rho_{p,q}, s)$-Carleson measure and prove (2.2.3). For $|a| \leq \frac{3}{4}$, (2.2.3) is obvious since the measure is necessarily finite. Let $\frac{3}{4} < |a| < 1$ and choose $\xi = a/|a|$. For any nonnegative integer k, let $r_k = 2^{k-1}(1-|a|)$, $k = 1, 2, \cdots, N$ and N the smallest integer such that $2^{N-2}(1-|a|) \geq 1$. Thus

$$\log_2\frac{4}{1-|a|} \leq N \leq 1+\log_2\frac{4}{1-|a|}. \tag{2.2.4}$$

Let $E_1 = Q_{r_1}(\xi)$ and $E_k = Q_{r_k}(\xi) - Q_{r_{k-1}}(\xi)$, $k \geq 2$. We have

$$\mu(E_k) \leq \mu(Q_{r_k}(\xi)) \leq \frac{C2^{(k-1)ns}(1-|a|)^{ns}}{(\log\frac{4}{2^{k-1}(1-|a|)})^p(\log\log\frac{e^4}{2^{k-1}(1-|a|)})^q}.$$

Moreover, if $k \geq 2$ and $z \in E_k$, then

$$\begin{aligned}
|1-\langle a,z\rangle| &= |1-|a|+|a|(1-\langle\xi,z\rangle)| \\
&\geq -(1-|a|)+|a||1-\langle\xi,z\rangle| \\
&\geq \frac{3}{4}2^{k-1}(1-|a|)-(1-|a|) \\
&\geq 2^{k-2}(1-|a|).
\end{aligned}$$

We also have for $z \in E_1$,

$$|1 - \langle z, a\rangle| \geq 1 - |a| > \frac{1}{2}(1 - |a|).$$

Using the above estimates, Hölder's inequality, the equivalence (2.2.4) and Lemma 2.2.1, we obtain

$$\begin{aligned}
\int_{\mathbb{B}^n} K_a^s(z)d\mu(z) &\leq \frac{C}{(1-|a|)^{ns}} \sum_{k=1}^{N} \frac{1}{2^{2nks}} \frac{r_k^{ns}}{(\log \frac{4}{r_k})^p (\log\log \frac{e^4}{r_k})^q} \\
&\lesssim \sum_{k=1}^{N} \frac{1}{2^{kns}} \frac{1}{(\log \frac{4}{2^{k-1}(1-|a|)})^p (\log\log \frac{e^4}{2^{k-1}(1-|a|)})^q} \\
&\lesssim \sum_{k=1}^{N} \frac{1}{2^{ks}} \frac{1}{(\log \frac{4}{2^{k-1}(1-|a|)})^p (\log\log \frac{e^4}{2^{k-1}(1-|a|)})^q} \\
&\lesssim \int_1^N \frac{1}{2^{ts}} \frac{1}{(\log \frac{4}{2^{t-1}(1-|a|)})^p (\log\log \frac{e^4}{2^{t-1}(1-|a|)})^q} dt \\
&\lesssim \left(\int_1^N \frac{1}{2^{ts}} \frac{1}{(\log \frac{4}{2^{t-1}(1-|a|)})^{p+q}} dt\right)^{\frac{p}{p+q}} \left(\int_1^N \frac{1}{2^{ts}} \frac{1}{(\log\log \frac{e^4}{2^{t-1}(1-|a|)})^{p+q}} dt\right)^{\frac{q}{p+q}} \\
&\leq \frac{C}{(\log \frac{4}{1-|a|})^p (\log\log \frac{e^4}{1-|a|})^q}.
\end{aligned}$$

This proves that (2.2.3) holds.

Now, suppose that (2.2.3) holds. For any $\xi \in \mathbb{S}^n$ and $0 < \delta < 1$, let $a = (1-\delta)\xi$. Then $1 - |a| = \delta$ and for $z \in Q_\delta(\xi)$, we have $K_a(z) \geq \frac{C}{\sigma(B_\delta(\xi))}$. Thus, we obtain easily that

$$\begin{aligned}
\infty &> C \gtrsim (\log \frac{4}{1-|a|})^p (\log\log \frac{e^4}{1-|a|})^q \int_{\mathbb{B}^n} K_a^s(z)d\mu(z) \\
&\gtrsim (\log \frac{4}{\delta})^p (\log\log \frac{e^4}{\delta})^q \int_{Q_\delta(\xi)} K_a^s(z)d\mu(z) \\
&\gtrsim \frac{(\log \frac{4}{\delta})^p (\log\log \frac{e^4}{\delta})^q}{(\sigma(B_\delta(\xi)))^s} \mu(Q_\delta(\xi)).
\end{aligned}$$

We conclude that μ is a $(\rho_{p,q}, s)$-Carleson measure. The proof is complete. □

The notation $A \lesssim B$ (resp. $A \gtrsim B$) means that there is a positive constant C such that $A \leq CB$ (resp. $A \geq CB$). The following is well-known (see also Lemma 2.2.4 below).

Lemma 2.2.3. *The following assertions hold.*

i) There exists a contant $C > 0$ such that for any $f \in BMOA$,

$$|f(z)| \leq C \log(\frac{4}{1-|z|})||f||_{BMOA}, \ \ z \in \mathbb{B}^n.$$

ii) The functions $f_a(z) = \log(\frac{4}{1-\langle z,a\rangle})$, $a \in \mathbb{B}^n$ *are in* $BMOA$ *with uniformly bounded norm.*

Lemma 2.2.4. *The following assertions hold.*

i) There exists a contant $C > 0$ *such that for any* $f \in LMOA$,

$$|f(z)| \leq C \log\log(\frac{e^4}{1-|z|})||f||_{LMOA}, \ \ z \in \mathbb{B}^n.$$

ii) The functions $f_a(z) = \log\log(\frac{e^4}{1-\langle z,a\rangle})$, $a \in \mathbb{B}^n$ *are in* $LMOA$ *with uniformly bounded norm.*

Proof. It is not hard to see that $LMOA$ is a subspace of $L\mathcal{B}$ with $||f||_{L\mathcal{B}} \leq C||f||_{LMOA}$ (see [117]). Thus, we only need to show that $i)$ holds for any $f \in L\mathcal{B}$.

For any analytic function f in $\mathbb{B}^n$, one easily has that

$$f(z) - f(0) = \int_0^1 \frac{Rf(tz)}{t} dt$$

for all $z \in \mathbb{B}^n$. It follows that there exists a contant $C > 0$ such that for any $f \in L\mathcal{B}$ and any $z \in \mathbb{B}^n$,

$$\begin{aligned} |f(z) - f(0)| &= |\int_0^1 \frac{Rf(tz)}{t} dt| \\ &\leq C||f||_{L\mathcal{B}} \int_0^1 \frac{|z|}{(1-|z|t)\log(\frac{e^4}{1-|z|t})} dt \\ &= C||f||_{L\mathcal{B}} (\log\log(\frac{e^4}{1-|z|}) - \log 4). \end{aligned}$$

This prove the pointwise estimate for all $f \in LMOA$.

Let us now prove that the functions $f_a(z) = \log\log(\frac{e^4}{1-\langle z,a\rangle})$ are uniformly in $LMOA$ or equivalently, by the characterization of [99], that the measures $d\mu_a(z) = |\nabla f_a(z)|^2(1-|z|^2)dV(z)$ are logarithmic-Carleson measures (that is a ρ-Carleson measures with $\rho(t) = \log^2(4/t)$) with uniform bound. For any $\xi \in \mathbb{S}^n$ and $0 < \delta < 1$, we set

$$I = \int_{|1-\langle z,\xi\rangle|<\delta} \frac{1-|z|^2}{|1-\langle a,z\rangle|^2 |\log(\frac{e^4}{1-\langle z,a\rangle})|^2} dV(z).$$

We have to show that $I \leq C\frac{\sigma(B_\delta(\xi))}{(\log\frac{4}{\delta})^2}$, where the constant $C > 0$ does not depend on the given $a \in \mathbb{B}^n$.

If $|1-\langle a,\xi\rangle| \geq 2\delta$, then for any $z \in \mathbb{B}^n$ with $|1-\langle z,\xi\rangle| < \delta$, $|1-\langle a,z\rangle| \geq \delta$. Thus,

$$I \leq \delta^{-2}(\log\frac{e^4}{\delta})^{-2} \int_{|1-\langle z,\xi\rangle|<\delta} (1-|z|^2)dV(z) \lesssim \frac{\sigma(B_\delta(\xi))}{(\log\frac{e^4}{\delta})^2}.$$

If $|1-\langle a,\xi\rangle| \le 2\delta$, we obtain

$$\begin{aligned}
I &\lesssim \int_{|1-\langle a,z\rangle|<3\delta} \frac{(1-|z|^2)}{|1-\langle z,a\rangle|^2\log^2\frac{e^4}{|1-\langle z,a\rangle|}}dV(z)\\
&\lesssim \sum_{j=0}^{\infty}\int_{3\delta 2^{-j-1}\le|1-\langle z,a\rangle|\le 3\delta 2^{-j}} \frac{(1-|z|^2)}{|1-\langle z,a\rangle|^2\log^2\frac{e^4}{|1-\langle z,a\rangle|}}dV(z)\\
&\lesssim \sum_{j=0}^{\infty} 2^{2(j+1)}\delta^{-2}(\log 2^j\frac{e^4}{\delta})^{-2}\int_{|1-\langle z,a\rangle|\le 3\delta 2^{-j}}(1-|z|^2)dV(z)\\
&\lesssim \frac{\delta^n}{(\log\frac{e^4}{\delta})^2}\sum_{j=0}^{\infty}2^{2(j+1)}2^{-j(n+2)} \lesssim \frac{\sigma(B_\delta(\xi))}{(\log\frac{4}{\delta})^2}.
\end{aligned}$$

The proof is complete. □

2.2.2 $\rho_{p,q}$- Carleson measures

In this subsection, we give and prove several equivalent definitions of $\rho_{p,q}$- Carleson measures. We first establish a useful lemma. Let φ_z be the involutive automorphism of $\mathbb{B}^n$ such that $\varphi_z(0)=z$ and $\varphi_z(z)=0$. We remark that for any $a, b,$ and $z\in\mathbb{B}^n$,

$$K_a(z)\cdot K_b(\varphi_a(z)) = K_{\varphi_a(b)}(z)$$

and

$$K_a(\varphi_a(z))\cdot K_a(z) = 1.$$

Lemma 2.2.5. *Let $0<s<\infty$ and let μ be a positive Borel measure on $\mathbb{B}^n$. Let*

$$d\mu_a(z) = \frac{d\mu(\varphi_a(z))}{K_a^s(z)}.$$

Then

$$\sup_{a\in\mathbb{B}^n}||\mu_a||_s \approx ||\mu||_s.$$

Proof: Using the previous remark, we obtain that

$$\begin{aligned}
\int_{\mathbb{B}^n} K_b^s(z)\frac{d\mu(\varphi_a(z))}{K_a^s(z)} &= \int_{\mathbb{B}^n} K_b^s(\varphi_a(w))\frac{d\mu(w)}{K_a^s(\varphi_a(w))}\\
&= \int_{\mathbb{B}^n} K_a^s(w)K_b^s(\varphi_a(w))d\mu(w)\\
&= \int_{\mathbb{B}^n} K_{\varphi_a(b)}^s(w)d\mu(w).
\end{aligned}$$

The conclusion follows by taking the supremum over $b\in\mathbb{B}^n$ and applying Theorem 2.2.2. □

Let us now recall the following equivalence for the norm of elements of BMOA space:

$$||f||_{BMOA} \approx \sup_{a\in\mathbb{B}^n} ||f \circ \varphi_a - f(a)||_p$$

for any $0 < p < \infty$ (see [117]).

Lemma 2.2.6. *Let $0 \leq p, q < \infty$ and let μ be a positive Borel measure on $\mathbb{B}^n$. Then the following conditions are equivalent.*

i) There exists a positive constant C_1 such that for any $0 < \delta < 1$ and any $\xi \in \mathbb{S}^n$,

$$\mu(Q_\delta(\xi)) \leq C_1 \frac{\sigma(B_\delta(\xi))}{(\log \frac{4}{\delta})^p (\log\log \frac{e^4}{\delta})^q}.$$

ii) There exists a positive constant C_2 such that

$$\sup_{a\in\mathbb{B}^n} (\log \frac{4}{1-|a|})^p (\log\log \frac{e^4}{1-|a|})^q \int_{\mathbb{B}^n} K_a(z) d\mu(z) \leq C_2 < \infty.$$

iii) There exists a positive constant C_3 such that for any $f \in BMOA$,

$$\sup_{a\in\mathbb{B}^n} (\log\log \frac{e^4}{1-|a|})^q \int_{\mathbb{B}^n} |f(z)|^p K_a(z) d\mu(z) \leq C_3 ||f||^p_{BMOA}.$$

iv) There exists a constant $C_4 > 0$ such that for any $f \in BMOA$ and any $g \in LMOA$,

$$\sup_{a\in\mathbb{B}^n} \int_{\mathbb{B}^n} |f(z)|^p |g(z)|^q K_a(z) d\mu(z) \leq C_4 ||f||^p_{BMOA} ||g||^q_{LMOA}.$$

Proof. The equivalence $i) \Leftrightarrow ii)$ follows from Theorem 2.2.2. We show that $ii) \Rightarrow iii) \Rightarrow iv) \Rightarrow i)$.

$ii) \Rightarrow iii)$: We first remark that $ii)$ implies that μ is a Carleson measure and so is $\frac{d\mu(\varphi_a(z))}{K_a(z)}$ for any fixed $a \in \mathbb{B}^n$ by Lemma 2.2.5.

Now, for any $f \in BMOA$, using Hölder's inequality we obtain

$$\begin{aligned}
\int_{\mathbb{B}^n} |f(z) - f(a)|^p K_a(z) d\mu(z) &\leq \left(\int_{\mathbb{B}^n} |f(z) - f(a)|^{p+q} K_a(z) d\mu(z)\right)^{\frac{p}{p+q}} \left(\int_{\mathbb{B}^n} K_a(z) d\mu(z)\right)^{\frac{q}{p+q}} \\
&\approx \left(\int_{\mathbb{B}^n} |fo\varphi_a(z) - f(a)|^{p+q} \frac{d\mu(\varphi_a(z))}{K_a(z)}\right)^{\frac{p}{p+q}} \left(\int_{\mathbb{B}^n} K_a(z) d\mu(z)\right)^{\frac{q}{p+q}} \\
&\leq C||\mu||^{p/(p+q)} ||fo\varphi_a - f(a)||^p_{p+q} \left(\int_{\mathbb{B}^n} K_a(z) d\mu(z)\right)^{\frac{q}{p+q}} \\
&\leq C||\mu||^{p/(p+q)} ||f||^p_{BMOA} \left(\int_{\mathbb{B}^n} K_a(z) d\mu(z)\right)^{\frac{q}{p+q}}.
\end{aligned}$$

It follows that

$$I_1 \leq C||\mu||^{p/(p+q)} ||f||^p_{BMOA} \left((\log\log \frac{e^4}{1-|a|})^{p+q} \int_{\mathbb{B}^n} K_a(z) d\mu(z)\right)^{\frac{q}{p+q}},$$

where

$$I_1 = (\log\log\frac{e^4}{1-|a|})^q \int_{\mathbb{B}^n} |f(z) - f(a)|^p K_a(z) d\mu(z).$$

It is also clear that $ii)$ implies that μ is a ρ-Carleson measure with

$$\rho(t) = (\log\log\frac{e^4}{t})^{p+q}, \quad t \in (0,1),$$

which is equivalent to saying there exists a constant $C > 0$ so that

$$(\log\log\frac{e^4}{1-|a|})^{p+q} \int_{\mathbb{B}^n} K_a(z) d\mu(z) \leq C < \infty.$$

We conclude that

$$I_1 = (\log\log\frac{e^4}{1-|a|^2})^q \int_{\mathbb{B}^n} |f(z) - f(a)|^p K_a(z) d\mu(z) \leq C||f||^p_{BMOA}. \tag{2.2.5}$$

Since $f \in BMOA$, we already know that there exists $C > 0$ so that

$$|f(a)| \leq C \log\frac{4}{1-|a|}||f||_{BMOA}.$$

Thus, setting

$$I_2 = (\log\log\frac{e^4}{1-|a|})^q \int_{\mathbb{B}^n} |f(a)|^p K_a(z) d\mu(z),$$

we obtain

$$I_2 \leq C(\log\frac{4}{1-|a|})^p(\log\log\frac{e^4}{1-|a|})^q||f||^p_{BMOA} \int_{\mathbb{B}^n} K_a(z) d\mu(z).$$

We conclude using Theorem 2.2.2 that

$$I_2 = (\log\log\frac{e^4}{1-|a|})^q \int_{\mathbb{B}^n} |f(a)|^p K_a(z) d\mu(z) \leq C||f||^p_{BMOA}, \tag{2.2.6}$$

where C is a constant independent of a. Finally, we obtain combining (2.2.5) and (2.2.6) that for any $a \in \mathbb{B}^n$,

$$\begin{aligned}(\log\log\frac{e^4}{1-|a|})^q \int_{\mathbb{B}^n} |f(z)|^p K_a(z) d\mu(z) &\leq 2^p(I_1 + I_2)\\ &\leq C_2||f||^p_{BMOA}.\end{aligned}$$

$iii) \Rightarrow iv)$: For any $f \in BMOA$, let $d\mu_f(z) = \frac{|f(z)|^p}{||f||^p_{BMOA}} d\mu(z)$. We would like to show that $iii)$ implies that there exists a positive constant C_4 such that for any $f \in BMOA$ and any $g \in LMOA$,

$$\sup_{a\in\mathbb{B}^n} \int_{\mathbb{B}^n} |g(z)|^q K_a(z) d\mu_f(z) \leq C_4||g||^q_{LMOA}.$$

We remark that $iii)$ implies in particular that for any $f \in BMOA$, the measure $d\mu_f$ is a Carleson measure with $||\mu_f|| \approx ||\mu||$. It follows easily as before that

$$\int_{\mathbb{B}^n} |g(z) - g(a)|^q K_a(z) d\mu_f(z) \leq C||\mu|| \times ||g||^q_{BMOA} \leq C||\mu|| \times ||g||^q_{LMOA}. \tag{2.2.7}$$

Now, using the pointwise estimate for $g \in LMOA$, we obtain

$$\int_{\mathbb{B}^n} |g(a)|^q K_a(z) d\mu_f(z) \quad \leq \quad C||g||^q_{LMOA} (\log\log \frac{e^4}{1-|a|})^q \int_{\mathbb{B}^n} K_a(z) d\mu_f(z).$$

It follows using $iii)$ that there exists $C > 0$ so that

$$\int_{\mathbb{B}^n} |g(a)|^q K_a(z) d\mu_f(z) \leq C||g||^q_{LMOA}. \tag{2.2.8}$$

Finally, using inequalities (2.2.7) and (2.2.8), we conclude that for any $a \in \mathbb{B}^n$,

$$\begin{aligned} \int_{\mathbb{B}^n} |f(z)|^p |g(z)|^q K_a(z) d\mu(z) \quad &\leq \quad 2^q \int_{\mathbb{B}^n} |f(z)|^p (|g(z) - g(a)|^q + |g(a)|^q) K_a(z) d\mu(z) \\ &\leq \quad C_2 ||f||^p_{BMOA} ||g(z)||^q_{LMOA}, \end{aligned}$$

which is $iv)$.

$iv) \Rightarrow i)$: For any $0 < \delta < 1$ and $\xi \in \mathbb{S}^n$, let $a = (1-\delta)\xi$. From $iv)$, we have in particular that there exists $C > 0$ so that for any $f \in BMOA$ and any $g \in LMOA$,

$$\int_{Q_\delta(\xi)} |f(z)|^p |g(z)|^q K_a(z) d\mu(z) \leq C||f||^p_{BMOA} ||g(z)||^q_{LMOA}.$$

We test the above inequality with $f(z) = f_a(z) = \log \frac{4}{1-\langle a,z \rangle}$ and $g(z) = g_a(z) = \log\log \frac{e^4}{1-\langle a,z \rangle}$ which are uniformly in $BMOA$ and $LMOA$ respectively. Remarking that for $z \in Q_\delta(\xi)$, $K_a(z) \geq \frac{C}{\sigma(B_\delta(\xi))}$, $\log \frac{4}{\delta} \leq |f_a(z)|$ and $\log\log \frac{e^4}{\delta} \leq |g_a(z)|$, we obtain

$$\begin{aligned} \frac{C}{\sigma(B_\delta(\xi))} (\log \frac{4}{1-|a|})^p (\log\log \frac{e^4}{1-|a|})^q \int_{Q_\delta(\xi)} d\mu(z) \quad &\leq \quad \int_{Q_\delta(\xi)} |f_a(z)|^p |g_a(z)|^q K_a(z) d\mu(z) \\ &\leq \quad C' < \infty. \end{aligned}$$

That is

$$\mu(Q_\delta(\xi)) \leq C \frac{\sigma(B_\delta(\xi))}{(\log \frac{4}{\delta})^p (\log\log \frac{e^4}{\delta})^q}.$$

The proof is complete. □

Taking $q = 0$ in the above lemma, we obtain the following corollary (see also [111]).

Corollary 2.2.7. *Let $0 \leq p < \infty$ and let μ be a positive Borel measure on $\mathbb{B}^n$. Then the following conditions are equivalent.*

i) There exists a positive constant C_1 such that for any $0<\delta<1$ and any $\xi \in \mathbb{S}^n$

$$\mu(Q_\delta(\xi)) \le C_1 \frac{\sigma(B_\delta(\xi))}{(\log \frac{4}{\delta})^p}.$$

ii) There exists a positive constant C_2 such that for any $f \in BMOA$,

$$\sup_{a\in\mathbb{B}^n} \int_{\mathbb{B}^n} |f(z)|^p K_a(z) d\mu(z) \le C_2 ||f||^p_{BMOA}.$$

Lemma 2.2.8. *Let $0 \le p, q < \infty$ and let μ be a positive Borel measure on $\mathbb{B}^n$. Then the following conditions are equivalent.*

i) There exists a positive constant C_1 such that for any $0<\delta<1$ and any $\xi \in \mathbb{S}^n$,

$$\mu(Q_\delta(\xi)) \le C_1 \frac{\sigma(B_\delta(\xi))}{(\log \frac{4}{\delta})^p (\log\log \frac{e^4}{\delta})^q}.$$

ii) There exists a positive constant C_2 such that for any $g \in LMOA$,

$$\sup_{a\in\mathbb{B}^n} (\log \frac{4}{1-|a|})^p \int_{\mathbb{B}^n} |g(z)|^q K_a(z) d\mu(z) \le C_2 ||g||^q_{LMOA}.$$

Proof. By Lemma 2.2.6, the assertion $i)$ is equivalent to saying there exists a constant $C>0$ such that for any $f \in BMOA$ and any $g \in LMOA$,

$$\sup_{a\in\mathbb{B}^n} \int_{\mathbb{B}^n} |f(z)|^p |g(z)|^q K_a(z) d\mu(z) \le C ||f||^p_{BMOA} ||g(z)||^q_{LMOA}.$$

It follows from Corollary 2.2.7 that the latter is equivalent to saying that there exists a positive constant C such that

$$\sup_{a\in\mathbb{B}^n} (\log \frac{4}{1-|a|})^p \int_{\mathbb{B}^n} K_a(z) d\mu_g(z) \le C < \infty,$$

where $d\mu_g(z) = \frac{|g(z)|^q}{||g(z)||^q_{LMOA}} d\mu(z)$. This proves $ii)$. The proof is complete. □

Theorem 2.2.9. *Let $0 \le p, q < \infty$ and let μ be a positive Borel measure on $\mathbb{B}^n$. Then the following conditions are equivalent.*

i) There is $C_1 > 0$ such that for any $\xi \in \mathbb{S}^n$ and $0<\delta<1$,

$$\mu(Q_\delta(\xi)) \le C_1 \frac{\sigma(B_\delta(\xi))}{(\log \frac{4}{\delta})^p (\log\log \frac{e^4}{\delta})^q}.$$

ii) There is $C_2 > 0$ such that

$$\sup_{a\in\mathbb{B}^n} (\log \frac{4}{1-|a|})^p (\log\log \frac{e^4}{1-|a|})^q \int_{\mathbb{B}^n} K_a(z) d\mu(z) \le C_2 < \infty.$$

iii) There is $C_3 > 0$ such that for any $f \in BMOA$,

$$\sup_{a\in\mathbb{B}^n}(\log\log\frac{e^4}{1-|a|})^q\int_{B_n}|f(z)|^pK_a(z)d\mu(z)\leq C_3||f||^p_{BMOA}.$$

iv) There is $C_4 > 0$ such that for any $g \in LMOA$,

$$\sup_{a\in\mathbb{B}^n}(\log\frac{4}{1-|a|})^p\int_{B_n}|g(z)|^qK_a(z)d\mu(z)\leq C_4||g||^q_{LMOA}.$$

v) There is $C_5 > 0$ such that for any $f \in BMOA$ and any $g \in LMOA$,

$$\sup_{a\in\mathbb{B}^n}\int_{\mathbb{B}^n}|f(z)|^p|g(z)|^qK_a(z)d\mu(z)\leq C_5||f||^p_{BMOA}||g||^q_{LMOA}.$$

vi) For $0 < r < \infty$, there is $C_6 > 0$ such that for any $f \in BMOA$ and any $g \in LMOA$ and any $h \in \mathcal{H}^r(\mathbb{B}^n)$,

$$\int_{\mathbb{B}^n}|f(z)|^p|g(z)|^q|h(z)|^rd\mu(z)\leq C_6||f||^p_{BMOA}||g||^q_{LMOA}||h||^r_r.$$

Proof. We already have from Lemma 2.2.6 and Lemma 2.2.8 that $i) \Leftrightarrow ii) \Leftrightarrow iii) \Leftrightarrow iv) \Leftrightarrow v)$. Let

$$d\mu_{f,g}(z)=\frac{|f(z)|^p|g(z)|^q}{||f(z)||^p_{BMOA}||g(z)||^q_{LMOA}}d\mu(z).$$

Then $v)$ is equivalent to saying that

$$\sup_{a\in\mathbb{B}^n}\int_{\mathbb{B}^n}K_a(z)d\mu_{f,g}<C_5.$$

By Theorem 2.1.8, this is equivalent to $vi)$. The proof is complete.

□

2.2.3 Some applications of $\rho_{p,q}$- Carleson measures

As first application of Theorem 2.2.9, we consider the Cesàro-type integral operator T_b defined by

$$T_b(f)(z)=\int_0^1 f(tz)Rb(tz)\frac{dt}{t},\quad b,\ f\in H(\mathbb{B}^n).$$

The characterization of the boundedness properties of T_b has been considered in [4], [5], [97] and [111] for the case of the unit disc and [103] for the case of the unit ball for some analytic function spaces. We first prove the following result on the boundedness of T_b on $LMOA$.

Corollary 2.2.10. *For $b \in H(\mathbb{B}^n)$, T_b is bounded on $LMOA$ if and only if*

$$\sup_{a\in\mathbb{B}^n}(\log\frac{4}{1-|a|})^2(\log\log\frac{e^4}{1-|a|})^2\int_{\mathbb{B}^n}|Rb(z)|^2(1-|z|^2)K_a(z)dV(z)<\infty. \tag{2.2.9}$$

Proof. We know from [99] that an analytic function b is in $LMOA$ if and only if $(1-|z|^2)|Rb(z)|^2dV(z)$ is a ρ-Carleson measure with $\rho(t) = (\log(4/t))^2$, which by Lemma 2.2.2 is equivalent to

$$\sup_{a\in\mathbb{B}^n}(\log\frac{4}{1-|a|})^2\int_{\mathbb{B}^n}|Rb(z)|^2(1-|z|^2)K_a(z)dV(z)<\infty.$$

It is not hard to see that

$$R[T_b(f)](z) = f(z)Rb(z).$$

It follows that T_b is bounded on $LMOA$ if and only if for any $f \in LMOA$,

$$\sup_{a\in\ \mathbb{B}^n}(\log\frac{4}{1-|a|})^2\int_{\mathbb{B}^n}|f(z)|^2|Rb(z)|^2(1-|z|^2)K_a(z)dV(z)<C||f||^2_{LMOA},$$

which by Theorem 2.2.9 is equivalent to saying that the measure $|Rb(z)|^2(1-|z|^2)dV(z)$ satisfies

$$\sup_{a\in\mathbb{B}^n}(\log\frac{2}{1-|a|})^2(\log\log\frac{e^4}{1-|a|})^2\int_{\mathbb{B}^n}|Rb(z)|^2(1-|z|^2)K_a(z)dV(z)<\infty.$$

The proof is complete. □

Corollary 2.2.11. *For $b \in H(\mathbb{B}^n)$, T_b is bounded from $LMOA$ to $BMOA$ if and only if*

$$\sup_{a\in\mathbb{B}^n}(\log\log\frac{e^4}{1-|a|})^2\int_{\mathbb{B}^n}|Rb(z)|^2(1-|z|^2)K_a(z)dV(z)<\infty. \tag{2.2.10}$$

Proof. We have already seen that an analytic b is in $BMOA$ if and only if $(1-|z|^2)|Rb(z)|^2dV(z)$ is a Carleson measure, that is

$$\sup_{a\in\mathbb{B}^n}\int_{\mathbb{B}^n}|Rb(z)|^2(1-|z|^2)K_a(z)dV(z)<\infty.$$

It follows that T_b is bounded from $LMOA$ to $BMOA$ if and only if for any $f \in LMOA$,

$$\sup_{a\in\ \mathbb{B}^n}\int_{\mathbb{B}^n}|f(z)|^2|Rb(z)|^2(1-|z|^2)K_a(z)dV(z)<C||f||^2_{LMOA}$$

which by Theorem 2.2.9 is equivalent to saying that the measure $|Rb(z)|^2(1-|z|^2)dV(z)$ satisfies

$$\sup_{a\in\mathbb{B}^n}(\log\log\frac{e^4}{1-|a|})^2\int_{\mathbb{B}^n}|Rb(z)|^2(1-|z|^2)K_a(z)dV(z)<\infty.$$

The proof is complete. □

We also obtain in the same way the following result.

Corollary 2.2.12. *For $b \in H(\mathbb{B}^n)$, T_b is bounded on $BMOA$ if and only if*

$$\sup_{a\in\mathbb{B}^n}(\log\frac{4}{1-|a|})^2\int_{\mathbb{B}^n}|Rb(z)|^2(1-|z|^2)K_a(z)dV(z)<\infty. \tag{2.2.11}$$

Our next application is about the pointwise multipliers on $LMOA$. Given two Banach spaces of analytic functions X and Y, we denote by $\mathcal{M}(X,Y)$ the space of multipliers from X to Y, that is

$$\mathcal{M}(X,Y) = \{f \in H(\mathbb{B}^n): \ f \cdot g \in Y \text{ for any } g \in X\}.$$

When $X = Y$, we just write $\mathcal{M}(X,X) = \mathcal{M}(X)$. The following lemma is an easy adaptation of [117, Lemma 3.20].

Lemma 2.2.13. *Suppose that X and Y are two Banach spaces of holomorphic functions. If X contains constant functions and each point evaluation is a bounded linear functional on both X and Y, then every pointwise multiplier from X to Y is in $\mathcal{H}^\infty(\mathbb{B}^n)$.*

We have the following characterization of $\mathcal{M}(LMOA)$ for the unit ball of $\mathbb{C}^n$

Corollary 2.2.14. *An analytic function f on $\mathbb{B}^n$ belongs to $\mathcal{M}(LMOA)$ if and only if $f \in \mathcal{H}^\infty(\mathbb{B}^n)$ and satisfies (2.2.9).*

Proof. Instead of using Lemma 2.2.13, we give a direct proof of the fact that any element in $\mathcal{M}(LMOA)$ is necessarily bounded. For this, we recall that for any $f \in LMOA$,

$$|f(z)| \leq C||f||_{LMOA} \log\log \frac{e^4}{1-|z|^2}.$$

Now, for any $a \in \mathbb{B}^n$, let $f_a(z) = \log\log(\frac{e^4}{1-\langle z,a\rangle})$. $f_a \in LMOA$ and $||f_a||_{LMOA} \leq C < \infty$.

It follows from these two facts that, if $f \in \mathcal{M}(LMOA)$, then $f \cdot f_a \in LMOA$ and for any $z \in \mathbb{B}^n$,

$$|f(z)f_a(z)| \leq C||f \cdot f_a||_{LMOA} \log\log \frac{e^4}{1-|z|^2}.$$

Taking $z = a$ in the above inequality, we obtain

$$|f(a)| \leq C||f \cdot f_a||_{LMOA} < C$$

where the constant C does not depend on $a \in \mathbb{B}^n$. That is $f \in H^\infty(\mathbb{B}^n)$.

If $f \in \mathcal{M}(LMOA)$, then for any $g \in LMOA$, the measure $|R(fg)(z)|^2(1-|z|^2)dV(z)$ is a logarithmic Carleson measure, or equivalently

$$I_f(g) \leq C||g||^2_{LMOA}, \tag{2.2.12}$$

where

$$I_f(g) = \sup_{a\in\mathbb{B}^n} (\log \frac{4}{1-|a|})^2 \int_{\mathbb{B}^n} |g(z)Rf(z) + f(z)Rg(z)|^2(1-|z|^2)K_a(z)dV(z).$$

Since $f \in H^\infty(\mathbb{B}^n)$ and $|Rg(z)|^2(1-|z|^2)dV(z)$ is a logarithmic Carleson measure,

$$\sup_{a\in\mathbb{B}^n}(\log\frac{4}{1-|a|})^2\int_{\mathbb{B}^n}|f(z)Rg(z)|^2(1-|z|^2)K_a(z)dV(z)\leq C||f||_\infty^2||g||_{LMOA}^2.$$

We deduce that if $f \in H^\infty(B_n)$, then (2.2.12) is equivalent to

$$\sup_{a\in\mathbb{B}^n}(\log\frac{4}{1-|a|})^2\int_{\mathbb{B}^n}|g(z)|^2|Rf(z)|^2(1-|z|^2)K_a(z)dV(z)\leq C||g||_{LMOA}^2,$$

which by Theorem 2.2.9 is equivalent to saying that $|Rf(z)|^2(1-|z|^2)dV(z)$ satisfies

$$\sup_{a\in\mathbb{B}^n}(\log\frac{2}{1-|a|})^2(\log\log\frac{e^4}{1-|a|})^2\int_{\mathbb{B}^n}|Rb(z)|^2(1-|z|^2)K_a(z)dV(z)<\infty.$$

The proof is complete. □

Similarly, we can prove the following results.

Corollary 2.2.15. *An analytic function f on $\mathbb{B}^n$ belongs to $\mathcal{M}(LMOA, BMOA)$ if and only if $f \in \mathcal{H}^\infty(\mathbb{B}^n)$ and satisfies (2.2.10).*

Corollary 2.2.16. *An analytic function f on $\mathbb{B}^n$ belongs to $\mathcal{M}(BMOA)$ if and only if $f \in \mathcal{H}^\infty(\mathbb{B}^n)$ and satisfies (2.2.11).*

The orthogonal projection of $L^2(\partial\mathbb{B}^n)$ onto $\mathcal{H}^2(\mathbb{B}^n)$ is called the Szegö projection and is denoted by P. It is given by

$$P(f)(z) = \int_{\partial\mathbb{B}^n} S(z,\xi)f(\xi)d\sigma(\xi), \tag{2.2.13}$$

where $S(z,\xi) = \frac{1}{(1-\langle z,\xi\rangle)^n}$ is the Szegö kernel on $\partial\mathbb{B}^n$. We denote as well by P its extension to $L^1(\partial\mathbb{B}^n)$.

For $b \in \mathcal{H}^2(\mathbb{B}^n)$, the small Hankel operator with symbol b is defined for f a bounded holomorphic function by

$$h_b(f) := P(b\overline{f}). \tag{2.2.14}$$

As last application, we prove that if $b \in LMOA$, then the Hankel operator h_b is bounded on $\mathcal{H}^1(\mathbb{B}^n)$. This extends the one dimensional result of [66] and [105].

Theorem 2.2.17. *The Hankel operator h_b extends into a bounded operator on $\mathcal{H}^1(\mathbb{B}^n)$ if $b \in LMOA$.*

Proof. Let $b \in LMOA$ or equivalently, such that $(1-|z|^2)|\nabla b(z)|^2dV(z)$ is a logarithmic Carleson measure. For $f \in \mathcal{H}^1(\mathbb{B}^n)$ and $g \in BMOA$, we want to estimate $|\langle h_b(f), g\rangle| =$

$|\langle b, fg\rangle|$. Applying Lemma 1.2.3 to $\langle b, fg\rangle$, it follows that we only need to estimate the following three integrals:

$$I_1 := \int_{\mathbb{B}^n} |f(z)||g(z)||b(z)|dV(z),$$

$$I_2 := \int_{\mathbb{B}^n} |f(z)| \left(|g(z)| + |\nabla g(z)|\right) |\nabla b(z)|(1-|z|^2)dV(z),$$

and

$$I_3 := \int_{\mathbb{B}^n} |g(z)||\nabla f(z)||\nabla b(z)|(1-|z|^2)dV(z).$$

For the first one, we observe that since g and b are in all $\mathcal{H}^p(\mathbb{B}^n)$, the estimate

$$|g(z)b(z)| \leq C(1-|z|^2)^{-1/2}$$

holds. It follows using the fact that the measure $(1-|z|^2)^{-1/2}dV(z)$ is a Carleson measure that

$$I_1 \leq C \int_{\mathcal{B}^n} |f(z)|(1-|z|^2)^{-1/2}dV(z) \leq C||f||_1.$$

For I_2, we use Cauchy-Schwarz inequality to obtain

$$I_2^2 \leq C \int_{\mathbb{B}^n} |f(z)| \left(|g(z)|^2 + |\nabla g(z)|^2\right) |(1-|z|^2)dV(z) \times \int_{\mathbb{B}^n} |f(z)||\nabla b(z)|^2(1-|z|^2)dV(z).$$

We conclude by using the fact that $|\nabla g(z)|^2(1-|z|^2)dV(z)$, $|\nabla b(z)|^2(1-|z|^2)dV(z)$ and $|g(z)|^2(1-|z|^2)dV(z)$ are Carleson measures.

The main point is the estimate of I_3. We first recall that, by the weak factorization theorem (see [36, 92]), any $f \in \mathcal{H}^1(\mathbb{B}^n)$ can be written as

$$f = \sum_j h_j l_j \quad \text{with} \quad \sum_j ||h_j||_2||l_j||_2 \leq C||f||_1.$$

Replacing f by this weak factorization, we are led to estimate a sum of terms as

$$J := \int_{\mathbb{B}^n} |g(z)||l(z)||\nabla h(z)||\nabla b(z)|(1-|z|^2)dV(z)$$

for l and h in $\mathcal{H}^2(\mathbb{B}^n)$. We recall that, for $h \in \mathcal{H}^2(\mathbb{B}^n)$,

$$\int_{\mathbb{B}^n} |\nabla h(z)|^2(1-|z|^2)dV(z) \leq C||h||_2.$$

Using this last inequality, Schwarz Inequality and Theorem 2.2.9, we obtain

$$\begin{aligned} J &\leq \left(\int_{\mathbb{B}^n} |\nabla h(z)|^2(1-|z|^2)dV(z)\right)^{1/2} \left(\int_{\mathcal{B}^n} |g(z)|^2|l(z)|^2|\nabla b(z)|^2(1-|z|^2)dV(z)\right)^{1/2} \\ &\leq C||g||_{BMOA}||l||_2||h||_2. \end{aligned}$$

This completes the proof of the theorem. □

Remark 2.2.18. The condition on the symbols of Hankel operators in Theorem 2.2.17 is also necessary (see [23]).

2.3 (ρ, s)-Carleson measures with $s > 1$

We consider in this section the case of (ρ, s)-Carleson measures when $s > 1$. Using Theorem 2.1.9 and the following equivalence for the norm of elements of the Bloch space $\mathcal{B}$:

$$||f||_{\mathcal{B}} \approx ||f \circ \varphi_a - f(a)||_{p,\alpha}, \ \ 0 < p < \infty \ \text{ and } \ \alpha > -1$$

(see [117]). We recall that the logarithmic Bloch space $L\mathcal{B}$ consists of holomorphic functions f such that

$$\|f\|_{L\mathcal{B}} = |f(0)| + \sup_{z \in \mathbb{B}^n} (1 - |z|^2)|Rf(z)| \log \frac{4}{1 - |z|^2} < \infty.$$

We can prove in the same way as Theorem 2.2.9, the following theorem.

Theorem 2.3.1. *Let $0 \leq p, q < \infty$, $1 < s < \infty$. Let μ be a positive Borel measure on $\mathbb{B}^n$. Then the following conditions are equivalent.*

i) There is $C_1 > 0$ such that for any $\xi \in \mathbb{S}^n$ and $0 < \delta < 1$,

$$\mu(Q_\delta(\xi)) \leq C_1 \frac{(\sigma(B_\delta(\xi)))^s}{(\log \frac{4}{\delta})^p (\log \log \frac{e^4}{\delta})^q}.$$

ii) There is $C_2 > 0$ such that

$$\sup_{a \in \mathbb{B}^n} (\log \frac{4}{1 - |a|})^p (\log \log \frac{e^4}{1 - |a|})^q \int_{\mathbb{B}^n} K_a(z)^s d\mu(z) \leq C_2 < \infty.$$

iii) There is $C_3 > 0$ such that for any $f \in \mathcal{B}$,

$$\sup_{a \in \mathbb{B}^n} (\log \log \frac{e^4}{1 - |a|})^q \int_{B_n} |f(z)|^p K_a^s(z) d\mu(z) \leq C_3 ||f||_{\mathcal{B}}^p.$$

iv) There is $C_4 > 0$ such that for any $g \in L\mathcal{B}$,

$$\sup_{a \in \mathbb{B}^n} (\log \frac{4}{1 - |a|})^p \int_{\mathbb{B}^n} |g(z)|^q K_a^s(z) d\mu(z) \leq C_4 ||g||_{L\mathcal{B}}^q.$$

v) There is $C_5 > 0$ such that for any $f \in \mathcal{B}$ and any $g \in L\mathcal{B}$,

$$\sup_{a \in \mathbb{B}^n} \int_{\mathbb{B}^n} |f(z)|^p |g(z)|^q K_a(z)^s d\mu(z) \leq C_5 ||f||_{\mathcal{B}}^p ||g||_{L\mathcal{B}}^q.$$

vi) For $0 < r < \infty$, there is $C_6 > 0$ such that for any $f \in \mathcal{B}$ and any $g \in L\mathcal{B}$ and any $h \in A^r_{ns-(n+1)}(\mathbb{B}^n)$,

$$\int_{\mathbb{B}^n} |f(z)|^p |g(z)|^q |h(z)|^r d\mu(z) \leq C_6 ||f||^p_{\mathcal{B}} ||g||^q_{L\mathcal{B}} ||h||^r_{ns-(n+1),r}.$$

We now move to applications of Theorem 2.3.1. We begin by considering the boundedness of the operator T_b on the logarithmic Bloch space $L\mathcal{B}$. We already know by Lemma 2.1.12 that a function $f \in H(\mathbb{B}^n)$ is in $L\mathcal{B}$ if and only if for any $s > 1$ the measure $(1-|z|^2)^{n(s-1)+1}|Rf(z)|^2 dV(z)$ is (ρ, s)-Carleson measure with $\rho(t) = (\log(4/t))^2$, or equivalently that

$$\sup_{a \in \mathbb{B}^n} (\log \frac{4}{1-|a|})^2 \int_{\mathbb{B}^n} K^s_a(z)(1-|z|^2)^{n(s-1)+1} |Rf(z)|^2 dV(z) < \infty.$$

We have the following corollary.

Corollary 2.3.2. *For $b \in H(\mathbb{B}^n)$, the operator T_b is bounded on $L\mathcal{B}$ if and only for any $s > 1$,*

$$\sup_{a \in \mathbb{B}^n} (\log \frac{4}{1-|a|})^2 (\log\log \frac{e^4}{1-|a|})^2 \int_{\mathbb{B}^n} |Rb(z)|^2 (1-|z|^2)^{n(s-1)+1} K^s_a(z) dV(z) < \infty. \quad (2.3.1)$$

Proof. Let

$$J_b(f) = \sup_{a \in \mathbb{B}^n} (\log \frac{4}{1-|a|})^2 \int_{\mathbb{B}^n} K^s_a(z)(1-|z|^2)^{n(s-1)+1} |f(z)|^2 |Rb(z)|^2 dV(z).$$

That T_b is bounded on $L\mathcal{B}$ is equivalent to saying there exists a constant $C > 0$ such that for any $s > 1$ and any $f \in L\mathcal{B}$,

$$J_b(f) < C||f||^2_{L\mathcal{B}}$$

which by Theorem 2.3.1 is equivalent to (2.3.1). □

Using Theorem 2.3.1 and the fact that any holomorphic function f belongs to $\mathcal{B}$ if and only if the measure $|Rf(z)|^2(1-|z|^2)^{n(s-1)+1} dV(z)$ is a s-Carleson measure for any $s > 1$, we can prove the following result in the same way.

Corollary 2.3.3. *For $b \in H(\mathbb{B}^n)$, the operator T_b is bounded from $L\mathcal{B}$ to $\mathcal{B}$ if and only for $s > 1$*

$$\sup_{a \in \mathbb{B}^n} (\log\log \frac{e^4}{1-|a|})^2 \int_{\mathbb{B}^n} |Rb(z)|^2 (1-|z|^2)^{n(s-1)+1} K^s_a(z) dV(z) < \infty. \quad (2.3.2)$$

The following well-known result (see for example [103]) follows in the same way.

Corollary 2.3.4. *For $b \in H(\mathbb{B}^n)$, the operator T_b is bounded on $\mathcal{B}$ if and only for $s > 1$*

$$\sup_{a\in\mathbb{B}^n} (\log \frac{4}{1-|a|})^2 \int_{\mathbb{B}^n} |Rb(z)|^2(1-|z|^2)^{n(s-1)+1} K_a^s(z) dV(z) < \infty. \tag{2.3.3}$$

We also obtain as in the previous section the following characterization of multipliers of Bloch-type spaces.

Corollary 2.3.5. *An analytic function f on $\mathbb{B}^n$ belongs to $\mathcal{M}(L\mathcal{B})$ if and only if $f \in \mathcal{H}^\infty(\mathbb{B}^n)$ and satisfies (2.3.1).*

Corollary 2.3.6. *An analytic function f on $\mathbb{B}^n$ belongs to $\mathcal{M}(L\mathcal{B}, \mathcal{B})$ if and only if $f \in H^\infty(\mathbb{B}^n)$ and satisfies (2.3.2).*

Corollary 2.3.7. *An analytic function f on $\mathbb{B}^n$ belongs to $\mathcal{M}(\mathcal{B})$ if and only if $f \in \mathcal{H}^\infty(\mathbb{B}^n)$ and satisfies (2.3.3).*

2.4 Some generalizations

We give some generalizations and their applications. The proofs here follow the same steps as in the two previous sections.

Theorem 2.4.1. *Let $0 \le p_1, p_2, q_1, q_2 < \infty$ and let μ be a positive Borel meaasure on $\mathbb{B}^n$. Then the following conditions are equivalent.*

i) There is $C_1 > 0$ such that for any $\xi \in \mathbb{S}^n$ and $0 < \delta < 1$,

$$\mu(Q_\delta(\xi)) \le C_1 \frac{\sigma(B_\delta(\xi))}{(\log \frac{4}{\delta})^{p_1+p_2} (\log\log \frac{e^4}{\delta})^{q_1+q_2}}.$$

ii) There is $C_2 > 0$ such that for any $f \in BMOA$ and any $g \in LMOA$

$$I(f,g) \le C_2 ||f||_{BMOA}^{p_1} ||g||_{LMOA}^{q_1},$$

where

$$I(f,g) \quad = \quad \sup_{a\in\mathbb{B}^n} (\log \frac{4}{1-|a|})^{p_2} (\log\log \frac{e^4}{1-|a|})^{q_2} \int_{\mathbb{B}^n} |f_1(z)|^{p_1} |g_1(z)|^{q_1} K_a(z) d\mu(z).$$

iii) There is $C_3 > 0$ such that for any $g \in LMOA$

$$I(g) \le C_3 ||g||_{LMOA}^{q_1},$$

where

$$I(g) = \sup_{a\in\mathbb{B}^n} (\log \frac{4}{1-|a|})^{p_1+p_2} (\log\log \frac{e^4}{1-|a|})^{q_2} \int_{\mathbb{B}^n} |g(z)|^{q_1} K_a(z) d\mu(z).$$

iv) There is $C_4 > 0$ such that for any $f \in BMOA$

$$I(f) \leq C_4 ||f||_{BMOA}^{p_1},$$

where

$$I(f) = \sup_{a\in\mathbb{B}^n} (\log \frac{4}{1-|a|})^{p_2} (\log\log \frac{e^4}{1-|a|})^{q_1+q_2} \int_{\mathbb{B}^n} |f(z)|^{p_1} K_a(z) d\mu(z).$$

Theorem 2.4.2. *Let $0 \leq p_1, p_2, q_1, q_2 < \infty$, let $1 < s < \infty$ and μ be a positive Borel measure on $\mathbb{B}^n$. Then the following conditions are equivalent.*

i) There is $C_1 > 0$ such that for any $\xi \in \mathbb{S}^n$ and $0 < \delta < 1$,

$$\mu(Q_\delta(\xi)) \leq C_1 \frac{(\sigma(B_\delta(\xi)))^s}{(\log \frac{4}{\delta})^{p_1+p_2} (\log\log \frac{e^4}{\delta})^{q_1+q_2}}.$$

ii) There is $C_2 > 0$ such that for any $f \in \mathcal{B}$ and any $g \in L\mathcal{B}$,

$$J(f,g) \leq C_2 ||f||_{\mathcal{B}}^{p_1} ||g||_{L\mathcal{B}}^{q_1},$$

where

$$J(f,g) \quad = \quad \sup_{a\in\mathbb{B}^n} (\log \frac{4}{1-|a|})^{p_2} (\log\log \frac{e^4}{1-|a|})^{q_2} \int_{\mathbb{B}^n} |f_1(z)|^{p_1} |g_1(z)|^{q_1} K_a^s(z) d\mu(z).$$

iii) There is $C_3 > 0$ such that for any $g \in L\mathcal{B}$,

$$\sup_{a\in\mathbb{B}^n} (\log \frac{4}{1-|a|})^{p_1+p_2} (\log\log \frac{e^4}{1-|a|})^{q_2} \int_{\mathbb{B}^n} |g(z)|^{q_1} K_a^s(z) d\mu(z) \leq C_3 ||g||_{L\mathcal{B}}^{q_1}.$$

iv) There is $C_4 > 0$ such that for any $f \in \mathcal{B}$

$$\sup_{a\in\mathbb{B}^n} (\log \frac{4}{1-|a|})^{p_2} (\log\log \frac{e^4}{1-|a|})^{q_1+q_2} \int_{\mathbb{B}^n} |f(z)|^{p_1} K_a^s(z) d\mu(z) \leq C_4 ||f||_{\mathcal{B}}^{p_1}.$$

Let $0 \leq p, q < \infty$. A function $f \in H(\mathbb{B}^n)$ belongs to $BMOA_{\rho_{p,q}}$ with $\rho_{p,q}(t) = (\log(4/t))^p (\log\log(e^4/t))^q$ if $f \in \mathcal{H}^1(\mathbb{B}^n)$ and there exists a constant $C > 0$ so that

$$\sup_{\substack{B=B_\delta(\xi)\\ \delta\in]0,1[,\xi\in\mathbb{S}^n}} \frac{(\log(4/\delta))^p (\log\log(e^4/\delta))^q}{\sigma(B)} \int_B |f - f_B| d\sigma \leq C.$$

By [99], a function f belongs to $BMOA_{\rho_{p,q}}$ if and only if $d\mu(z) = (1-|z|^2)|\nabla f(z)|^2 dV(z)$ is a $\rho_{p,q}^2$- Carleson measure. The following corollaries can be proved as in the previous sections.

Corollary 2.4.3. *Let $0 \le p, q < \infty$. Given an analytic function b, the operator T_b is bounded from $LMOA$ to $BMOA_{\rho_{p,q}}$ if and only if*

$$\sup_{a\in\mathbb{B}^n}(\log\frac{4}{1-|a|})^{2p}(\log\log\frac{e^4}{1-|a|})^{2q+2}\int_{\mathbb{B}^n}|Rb(z)|^2(1-|z|^2)K_a(z)dV(z)<\infty. \quad (2.4.1)$$

Corollary 2.4.4. *Let $0 \le p, q < \infty$. Given an analytic function b, the operator T_b is bounded from $BMOA$ to $BMOA_{\rho_{p,q}}$ if and only if*

$$\sup_{a\in\mathbb{B}^n}(\log\frac{4}{1-|a|})^{2p+2}(\log\log\frac{e^4}{1-|a|})^{2q}\int_{\mathbb{B}^n}|Rb(z)|^2(1-|z|^2)K_a(z)dV(z)<\infty. \quad (2.4.2)$$

In particular, we have the following.

Corollary 2.4.5. *Given an analytic function b, the operator T_b is bounded from $BMOA$ to $LMOA$ if and only if*

$$\sup_{a\in\mathbb{B}^n}(\log\frac{4}{1-|a|})^{4}\int_{\mathbb{B}^n}|Rb(z)|^2(1-|z|^2)K_a(z)dV(z)<\infty.$$

Let us now move to applications of Theorem 2.4.2. The following two corollaries can be proved exactly as before.

Corollary 2.4.6. *Let $0 \le p, q < \infty$, $\alpha > 0$ and $b \in H(\mathbb{B}^n)$. Then the following conditions are equivalent.*

(a) T_b is bounded from $L\mathcal{B}$ to $\mathcal{B}^{p,q}_\alpha$.

(b) For any $s > 1$,

$$\sup_{a\in B_n}(\log\frac{4}{1-|a|})^{2p}(\log\log\frac{e^4}{1-|a|})^{2q+2}\int_{\mathbb{B}^n}|Rb(z)|^2(1-|z|^2)^{n(s-1)+2\alpha-1}K^s_a(z)dV(z)<\infty. \quad (2.4.3)$$

Corollary 2.4.7. *Let $0 \le p, q < \infty$, $\alpha > 0$ and $b \in H(\mathbb{B}^n)$. Then the following conditions are equivalent.*

(a) T_b is bounded from $\mathcal{B}$ to $\mathcal{B}^{p,q}_\alpha$.

(b) For any $s > 1$,

$$\sup_{a\in B_n}(\log\frac{4}{1-|a|})^{2p+2}(\log\log\frac{e^4}{1-|a|})^{2q}\int_{\mathbb{B}^n}|Rb(z)|^2(1-|z|^2)^{n(s-1)+2\alpha-1}K^s_a(z)dV(z)<\infty. \quad (2.4.4)$$

In particular, we have the following.

Corollary 2.4.8. *Given $b \in H(\mathbb{B}^n)$, the operator T_b is bounded from $\mathcal{B}$ to $L\mathcal{B}$ if and only if for any $s > 1$,*

$$\sup_{a\in B_n}(\log\frac{4}{1-|a|})^{4}\int_{\mathbb{B}^n}|Rb(z)|^2(1-|z|^2)^{n(s-1)+2\alpha-1}K^s_a(z)dV(z)<\infty.$$

Part II

Logarithmic mean oscillations on the bidisc

Chapter 3

Bi-parameter paraproducts on the bidisc

We introduce the space of functions of dyadic logarithmic mean oscillation on the bidisc and use it to characterize boundedness of dyadic bi-parameter paraproducts on the product space of function of bounded dyadic mean oscillation, $\mathrm{BMO}^d(\mathbb{T}^2)$. The idea of our proof consists in writing our operators as a sum of localized operators satisfying some good estimates and then applying Cotlar's lemma.

3.1 Introduction

The paraproducts $\pi(f,b)$ are bilinear (multilinear) operators representing a class of operators which can be view in some sense as "half product" or "renormalized product" ([71]). They first appeared in the work of Bony ([26]) in relation with nonlinear differential equations. Since then they have appeared as important tools in Harmonic Analysis. Their importance can be illustrated from the $T(1)$ theorem of David and Journé [69] which claims that many singular integral operators T can be written as $T = S + \pi + (\pi)^*$, where S is an almost translation invariant (or convolution) operator.

The study of one parameter and multi-parameter paraproducts has attracted a lot of attention in very recent years [21, 54, 68, 71–73, 75, 76, 78, 84] with application to various problems in Analysis. In [75, 76], their characterization appears as an important step in the study of of a multi-parameter version the Coifman-Meyer theorem ([33, 34]). In [87] the authors also used the properties of paraproducts to characterize the Hankel operators of Schatten class while in [72], they are used in the study of boundedness of commutators with Riesz potentials. The study of composition of (Haar) paraproducts in [20] is applied to the two weights problem, i.e for which pair of weights (u, v) is the Hilbert transform bounded from $L^2(du)$ to $L^2(dv)$?

One of the situations considered in [73] is the boundedness of the following multilinear map:

$$\pi : L^{p_1} \times \cdots \times L^{p_n} \to L^r, \quad 1 < p_j \leq \infty$$
$$\sum_{j=1}^{n} \frac{1}{p_j} = \frac{1}{r}.$$

It is understood that for some appropriate paraproducts π and in the limit case, i.e when for example p_1 is 1 or ∞, the space L^{p_1} can be replaced by the real Hardy space H^1_{Re} ($p_1 = 1$) or its dual space BMO ($p_1 = \infty$). Let us remark that paraproducts have been used in [21] to provided alternative characterizations of the Chang-Fefferman BMO space. Restricting ourself to the bi-parameter case, an interesting situation which is still open for general paraproducts is the following

$$\pi : BMO \times L^2 \to L^2.$$

In this chapter, this last case is considered for some special Haar dyadic paraproduts in the bitorus. The motivation for considering this case in the product setting is related to the study of some other operators, such as the multiplication operators for BMO considered in the next chapter or, as we expect Hankel operators in the limit case H^1_{Re}.

3.2 Settings and definitions

3.2.1 The one dimensional case

We recall in this subsection some basic facts about dyadic Harmonic Analysis in the unit circle. Most of our statements are from [81].

Let $\mathbb{T}$ denote the unit circle which we identify with the interval $[0, 1)$. We recall that for $1 \leq p < \infty$, a function f belongs to the Lebesgue space $L^p(\mathbb{T}) = L^p(\mathbb{T}, dt)$, if f satisfies the following integrability condition:

$$||f||_p^p := \int_{\mathbb{T}} |f(t)|^p dt < \infty.$$

The inner product on the Hilbert space $L^2(\mathbb{T})$ is given by

$$\langle f, g \rangle := \int_{\mathbb{T}} f(t)\overline{g(t)} dt.$$

A dyadic interval is any interval of the form $[k2^{-j},(k+1)2^{-j})$ with j,k integers. We write $\mathcal{D}$ for the set of dyadic intervals in $\mathbb{T}$. Given an integer k, the k-th generation of dyadic intervals is defined by

$$\mathcal{D}_k = \{I \in \mathcal{D} : |I| = 2^{-k}\}.$$

Remark 3.2.1. Let us remark that

- Given two intervals I and J in $\mathcal{D}$, they are either disjoint or one is contained in the other.
- Each interval is in a unique generation $\mathcal{D}_k$ and there are exactly two subintervals of I in the next generation $\mathcal{D}_{k+1}$ called the children of I: the right half I^+ and the left half I^-. Moreover, $I = I^+ \bigcup I^-$.
- For every interval I in $\mathcal{D}_k$ there exists exactly one interval $\tilde{I}$ in $\mathcal{D}_{k-1}$ such that $I \subset \tilde{I}$. $\tilde{I}$ is called the parent interval of I

Let h_I denote the Haar wavelet adapted to the dyadic interval I,

$$h_I = |I|^{-1/2}(\chi_{I^+} - \chi_{I^-})$$

where I^+ and I^- are the right and left halves of I, respectively and χ_I is the characteristic function of I:

$$\chi_I(t) = \begin{cases} 1 & \text{if} \quad t \in I \\ 0 & \text{otherwise} \end{cases}$$

The set of functions $\{h_I : I \in \mathcal{D}\} \cup \{\chi_{[0,1]}\}$ forms an orthonormal basis for $L^2([0,1])$.

Remark 3.2.2. Because of the application we have in mind and for simplicity, we suppose in this chapter that our functions have mean zero over $[0,1]$. This has the advantage of reducing the set of constant functions to $\{0\}$. We denote by $L^2_0(\mathbb{T})$ the subset of $L^2(\mathbb{T})$ corresponding to such functions. Then any f in $L^2_0(\mathbb{T})$ has the expansion:

$$f = \sum_{I \in \mathcal{D}} \langle f, h_I \rangle h_I = \sum_{I \in \mathcal{D}} f_I h_I$$

(see for example [81]).

We will be writing $m_I f = \frac{1}{|I|}\int_I f(t)dt$ for the mean of the function f over the dyadic interval I.

The space of function of dyadic bounded mean oscillation in $\mathbb{T}$, $BMO^d(\mathbb{T})$, is the space of all functions $f \in L^2(\mathbb{T})$ such that

$$||f||_*^2 := \sup_{I\in\mathcal{D}} \frac{1}{|I|}\int_I |f - m_I f|^2 dt < \infty. \tag{3.2.1}$$

A norm on $BMO^d(\mathbb{T})$ is given by

$$||f||_{BMO^d}^2 := ||f||_2^2 + ||f||_*^2.$$

It follows using the Haar expansion that

$$||f||_*^2 := \sup_{J\in\mathcal{D}} \frac{1}{|J|}\sum_{I\subseteq J}|f_I|^2 = \sup_{J\subset\mathbb{T}} \frac{1}{|J|}||P_J f||_2^2 < \infty, \tag{3.2.2}$$

where the supremum is taken over all dyadic interval $J \subset \mathbb{T}$ and P_J is the orthogonal projection on the subspace spanned by Haar functions h_I, $I \in \mathcal{D}$ and $I \subseteq J$, i.e.

$$P_J(f) = \sum_{I\subseteq J,\ I\in\mathcal{D}} f_I h_I.$$

Remark 3.2.3. The usual definition of the space of function of bounded mean oscillation in the unit circle uses the power 1 in (3.2.1). The John-Nirenberg's theorem then allows us to use any power $1 \leq p < \infty$ (see [49, 67]).

The space $BMO^d(\mathbb{T})$ is the dual space of the dyadic Hardy space $H_d^1(\mathbb{T})$ defined in terms of the dyadic square function

$$\mathcal{S}(f)(x) = (\sum_{x\in I\in\mathcal{D}} \frac{|\langle f, h_I\rangle|^2}{|I|})^{1/2}.$$

That means,

$$H_d^1(\mathbb{T}) = \{f \in L^1(\mathbb{T}) : \mathcal{S}f \in L^1(\mathbb{T})\}$$

(see [81]).

Given f, g with finite Haar expansion, we have

$$fg = \pi_g(f) + (\pi_{\overline{g}})^*(f) + \pi_f(g),$$

where π_b is the dyadic paraproduct with symbol b defined on $L^2(\mathbb{T})$ by

$$\pi_b(f) = \sum_{I\in\mathcal{D}} b_I m_I f h_I$$

and its adjoint $(\pi_b)^* = \Delta_{\overline{b}}$ is given by

$$\Delta_b(f) = \sum_{I\in\mathcal{D}} b_I f_I \frac{\chi_I}{|I|}.$$

It is well-known that π_b is bounded on $L^p(\mathbb{T})$ if and only $b \in BMO^d(\mathbb{T})$ (see [81]).

3.2.2 The product setting case

Let $\mathbb{T}^2$ denote the product of the unit circle by itself. We recall that $\mathcal{D}$ is the set of dyadic intervals in $\mathbb{T}$ and we denote by $\mathcal{R}$ the set of all dyadic rectangles $R = I \times J$, I and J in $\mathcal{D}$. For any rectangle $R \in \mathcal{R}$, the product Haar wavelet adapted to $R = I \times J$ is defined by $h_R(t,s) = h_I(t)h_J(s)$. These wavelets form an orthonormal basis of $L_0^2(\mathbb{T}^2)$:

$$f = \sum_{R\in\mathcal{R}} \langle f, h_R\rangle h_R = \sum_{R\in\mathcal{R}} f_R h_R.$$

We will be writing $m_R f$ for the mean of $f \in L^2(\mathbb{T}^2)$ over the dyadic rectangle R. We also use the notation $f_I(s) = \langle f(.,s), h_I\rangle$, $m_I f(s) = \frac{1}{|I|}\int_I f(t,s)dt$, $f_I(t) = \langle f(t,.), h_J\rangle$ and $m_J f(t) = \frac{1}{|J|}\int_J f(t,s)ds$.

For any $f \in L^2(\mathbb{T}^2)$, we use the notations

$$P_I f = \sum_{R'=I'\times J'\in\mathcal{R}, I'\subseteq I} h_{R'} f_{R'}$$

and

$$P_J f = \sum_{R'=I'\times J'\in\mathcal{R}, J'\subseteq J} h_{R'} f_{R'}.$$

This means that given $I \in \mathcal{D}$, P_I is the orthogonal projection on the subspace spanned by the Haar functions $h_{R'} = h_{I',J'}$, $R' = I' \times J' \in \mathcal{R}$, $I' \subseteq I$.

There are several notions of bounded mean oscillation in the product setting. We are interested here only in the Chang-Fefferman version of [30] . The space of functions of dyadic bounded mean oscillation in $\mathbb{T}^2$, BMO$^d(\mathbb{T}^2)$, is the space of all function $f \in L^2(\mathbb{T}^2)$ such that

$$||f||^2_{BMO^d} := \sup_{\Omega\subset\mathbb{T}^2} \frac{1}{|\Omega|} \sum_{R\subseteq\Omega} |f_R|^2 = \sup_{\Omega\subset\mathbb{T}^2} \frac{1}{|\Omega|} ||P_\Omega f||_2^2 < \infty, \tag{3.2.3}$$

where the supremum is taken over all open sets $\Omega \subset \mathbb{T}^2$ and P_Ω the orthogonal projection on the subspace spanned by Haar functions h_R, $R \in \mathcal{R}$ and $R \subseteq \Omega$. It is well-known (see [30]) that $\mathrm{BMO}^d(\mathbb{T}^2)$ is the dual space of the dyadic product Hardy space $H^1_d(\mathbb{T}^2)$ defined in terms of the dyadic square function

$$\mathcal{S}(f)(t,s) = \Big(\sum_{(t,s)\in R\in\mathcal{R}} \frac{|\langle f, h_R\rangle|^2}{|R|}\Big)^{1/2}.$$

That is,

$$H^1_d(\mathbb{T}^2) = \{f \in L^1(\mathbb{T}^2) : \mathcal{S}f \in L^1(\mathbb{T}^2)\}.$$

As usual, we define the (i,j)th generation of dyadic rectangles,

$$\mathcal{R}_{i,j} = \{I \times J \in \mathcal{R} : |I| = 2^{-i}, |J| = 2^{-j}\},$$

the product Haar martingale difference,

$$\Delta_{i,j} f = \sum_{R\in\mathcal{R}_{i,j}} \langle f, h_R\rangle h_R,$$

the expectations

$$E_{i,j} f = \sum_{k<i, l<j} \Delta_{k,l} f,$$
$$E^{(1)}_i f = \sum_{k<i, l,k\in\mathbb{N}_0} \Delta_{k,l} f,$$
$$E^{(2)}_j f = \sum_{l,k\in\mathbb{N}_0, l<j} \Delta_{k,l} f,$$

for $f \in L^2(\mathbb{T})$, $i,j \in \mathbb{N}$. We will need also need the operators on $L^2(\mathbb{T}^2)$ given by

$$Q_{i,j} f = \sum_{k\geq i, l\geq j} \Delta_{k,l} f$$
$$Q^{(1)}_i f = \sum_{k\geq i, l,k\in\mathbb{N}_0} \Delta_{k,l} f$$
$$Q^{(2)}_j f = \sum_{l,k\in\mathbb{N}_0, l\geq j} \Delta_{k,l} f.$$

Note that $Q_{i,j}$ is not the orthogonal complement of the expectation $E_{i,j}$. In fact we have the decomposition

$$f = E_{i,j} f + E^{(1)}_i Q^{(2)}_j f + E^{(2)}_i Q^{(1)}_j f + Q_{i,j} f. \tag{3.2.4}$$

The one dimensional paraproduct π and its adjoint Δ can be combined to obtain various two dimension paraproducts. In particular, we write for $\varphi, f \in L^2(\mathbb{T}^2)$,

$$\pi_{\pi_\varphi} f = \pi^{(1,2)}_\varphi f = \sum_{i,j\geq 0} (\Delta_{i,j}\varphi)(E_{i,j} f) = \sum_{R\in\mathcal{R}} h_R \varphi_R m_R f.$$

We study below boundedness criteria of such operators on the product space $BMO^d(\mathbb{T}^2)$.

3.2.3 An example of a function in $\mathrm{BMO}^d(\mathbb{T}^2)$

We show how to obtain for each rectangle $R = I \times J \subset \mathbb{T}^2$, a function $f^R \in BMO^d(\mathbb{T}^2)$ such that

$$f^R \geq C \log \frac{4}{|I|} \log \frac{4}{|J|} \chi_I \chi_J. \tag{3.2.5}$$

Here the constant $C > 0$ does not depend on R. In fact, we have the following lemma in [21].

Lemma 3.2.4. $\mathrm{BMO}^d(\mathbb{T}) \bigotimes \mathrm{BMO}^d(\mathbb{T}) \subseteq BMO^d(\mathbb{T}^2)$.

The above lemma says that given f and g in $\mathrm{BMO}^d(\mathbb{T})$, their tensor product defined by $b(t,s) := f(t)b(s)$ is in $\mathrm{BMO}^d(\mathbb{T}^2)$. Thus, to obtain an example of element of $\mathrm{BMO}^d(\mathbb{T}^2)$ satisfying (3.2.5), we only need to find for any interval J a function f^J in the one dimensional space $\mathrm{BMO}^d(\mathbb{T})$ which satisfies the estimate

$$f^J \geq C \log \frac{4}{|J|}$$

where the constant $C > 0$ does not depend on J.

Let J be a fixed interval in $\mathbb{T}$. Let $J_0 = J$ and J_k be the intervals in $\mathbb{T}$ with the same center as J and such that $|J_k| = 2^k|J|$, here $k = 1, 2, \cdots, N-1$ and N is the smallest integer such that $2^N|J| \geq 1$. We define $J_N = \mathbb{T}$. Thus,

$$N + 2 \geq \log_2 \frac{4}{|J|}.$$

Remark that the intervals J_k are not necessarily dyadic. We define $U_0 = J_0 = J$, $U_k = J_k \setminus J_{k-1}$, for $k = 1, \cdots, N$. Now consider the function f^J defined on $\mathbb{T}$ by

$$f^J = \sum_{k=0}^{N} (N + 2 - k) \chi_{U_k}. \tag{3.2.6}$$

Clearly,

$$f^J(t) \geq N + 2 \geq \log_2 \frac{4}{|J|} \text{ for all } t \in J.$$

Lemma 3.2.5. *For each interval $J \subset \mathbb{T}$, the function f^J defined by (3.2.6) belongs to $BMO(\mathbb{T})$.*

Proof. We first estimate the L^2-norm of f^J. We have

$$\begin{aligned} ||f^J||_2^2 = \sum_{k=0}^{N}(N+2-k)^2|U_k| = \sum_{k=2}^{N+2} k^2|J_{N+2-k}| &\leq \sum_{k=1}^{N+2} k^2 2^{N+2-k}|J| \\ &\leq \sum_{k=1}^{N+2} k^2 2^{N+2-k} 2^{1-N} \\ &= 8\sum_{k=1}^{N+2} k^2 2^{-k}. \end{aligned}$$

It is clear that the last sum in the above equalities is finite and so $f^J \in L^2(\mathbb{T})$.

For any interval $I \in \mathbb{T}$, let $m \in 0, \cdots, N+1$ be minimal such that $I \cap U_m \neq \emptyset$, and $l \in 0, \cdots, N+1$ be maximal such that $I \cap U_{m+l} \neq \emptyset$. Let us estimate the length of $I \cap U_j$ for any $m \leq j \leq m+l$. If $l = 0$ then $I \cap U_m = I$ and there is nothing to say. If $l = 1$ then $|I \cap U_m| \leq |I|$ and $|I \cap U_{m+1}| \leq |I|$. Next, we consider the case $l \geq 2$. We remark that in this case, at least half of U_j is contained in I for any $m < j < m+l$, thus we have $|I \cap U_j| \leq 2\frac{1}{2^{m+l-j-1}}|I|$. Finally, we have $|I \cap U_m| \leq |I|$. Thus,

$$\begin{aligned} \frac{1}{|I|}\int_I |f^J - (N+2-m-l)|dt &= \frac{1}{|I|}\int_I |\sum_{k=m}^{m+l}(m+l-k)\chi_{U_k}|dt \\ &\leq \frac{1}{|I|}\sum_{k=m}^{m+l}(m+l-k)|I \cap U_k| \\ &\leq 4\frac{1}{|I|}\sum_{k=m}^{m+l}(m+l-k)2^{-m-l+k}|I| \\ &= 4\sum_{k=0}^{k=l}\frac{k}{2^k} \leq 8. \end{aligned}$$

Thus, for each interval $J \in \mathcal{D}$, the function f^J given by (3.2.6) belongs to $\mathrm{BMO}(\mathbb{T})$ and there exists a positive constant C independent of J such that $||f^J||_{\mathrm{BMO}} \leq C$. The proof is complete. $\square$

3.3 Boundedness of paraproducts on $\mathrm{BMO}^d(\mathbb{T}^2)$

Given two function f and g in $L_0^2(\mathbb{T}^2)$ with finite Haar expansion, the pointwise product $f \cdot g$ can be written as the following

$$fg = \pi_{\pi_g} f + \Delta_{\Delta_g} f + \pi_{\Delta_g} f + \Delta_{\pi_g} f + R_{\Delta_g} f + \Delta_{R_g} f + R_{\pi_g} f + \pi_{R_g} f + R_{R_g} f.$$

The nine terms correspond to the products $\langle M_\varphi h_I(s)h_J(t), h_{I'}(s)h_{J'}(t)\rangle$ for $I' \subset I$, $I' = I$, $I' \supset I$, $J' \subset J$, $J' = J$, $J' \supset J$. The first four operators above can be seen as compositions of the one dimensional paraproduct π and its adjoint Δ. They are defined on $L^2(\mathbb{T}^2)$ by the following formulas.

$$\pi_{\pi_\varphi} f = \pi_\varphi f = \sum_{i,j\geq 0} (\Delta_{i,j}\varphi)(E_{i,j}f) = \sum_{R\in\mathcal{R}} h_R \varphi_R m_R f,$$

$$\Delta_{\Delta_\varphi} f = \sum_{R\in\mathcal{R}} \frac{\chi_R}{|R|}\varphi_R f_R,$$

$$\Delta_{\pi_\varphi} f = \sum_{I\times J\in\mathcal{R}} \frac{\chi_I(s)}{|I|} h_J(t)\varphi_{I\times J} m_J f_I,$$

$$\pi_{\Delta_\varphi} f = \sum_{I\times J\in\mathcal{R}} h_I(s)\frac{\chi_J(t)}{|J|}\varphi_{I\times J} m_I f_J.$$

We refer to the next chapter for the definition of the remaining five terms.

It is a consequence of the Chang's generalization of Carleson Embedding Theorem (see [29]) that π_{π_φ} is bounded on $L^2(\mathbb{T}^2)$ if and only if $\varphi \in \mathrm{BMO}^d(\mathbb{T}^2)$. The boundedness of the operators π_{Δ_φ} and Δ_{π_φ} has been studied in [21] and [84]. The following results can be found in [21, 71].

Proposition 3.3.1. *Let $\varphi \in L^2(\mathbb{T}^2)$. Then the following assertions hold.*

a) The operators π_{π_φ} and Δ_{Δ_φ} are bounded on $L^2(\mathbb{T}^2)$ if and only if $\varphi \in \mathrm{BMO}^d(\mathbb{T}^2)$. Moreover,

$$\|\pi_{\pi_\varphi}\|_{L^2\to L^2} \approx \|\varphi\|_{\mathrm{BMO}^d}.$$

b) If $\varphi \in \mathrm{BMO}^d(\mathbb{T}^2)$, then both π_{Δ_φ} and Δ_{π_φ} are bounded on $L^2(\mathbb{T}^2)$.

In this section, we characterize those symbols for which they above operators extend as bounded operators on $\mathrm{BMO}^d(\mathbb{T}^2)$. For this, we introduce the following notions of function of dyadic logarithmic oscillation in product setting.

Definition 3.3.2. Let $\varphi \in L^2(\mathbb{T}^2)$.

- We say that $\varphi \in \mathrm{LMO}^d(\mathbb{T}^2)$, if there exists $C > 0$ with

$$\|Q_{i,j}\varphi\|_{\mathrm{BMO}^d(\mathbb{T}^2)} \leq C\frac{1}{ij}$$

  for all i, j.

- We say that $\varphi \in \mathrm{LMO}^d_1(\mathbb{T}^2)$, if there exists $C > 0$ with

$$\|Q^{(1)}_i\varphi\|_{\mathrm{BMO}^d} \leq C\frac{1}{i}$$

  for all $i \in \mathbb{N}$.

- We say that $\varphi \in \mathrm{LMO}_2^d(\mathbb{T}^2)$, if there exists $C > 0$ with
$$\|Q_j^{(2)}\varphi\|_{\mathrm{BMO}^d} \leq C\frac{1}{j}$$
for all $j \in \mathbb{N}$.

The infimum of such constants is denoted by $\|\varphi\|_{\mathrm{LMO}^d}$,$\|\varphi\|_{\mathrm{LMO}_1^d}$, $\|\varphi\|_{\mathrm{LMO}_2^d}$, respectively.

Remark 3.3.3. An alternative characterization of $\mathrm{LMO}^d(\mathbb{T}^2)$, which is closer in spirit to the one-parameter case, is the following: Let $\varphi \in L^2(\mathbb{T}^2)$. Then

$\varphi \in \mathrm{LMO}^d(\mathbb{T}^2)$, if and only if there exists $C > 0$ such that for each dyadic rectangle $R = I \times J$ and each open set $\Omega \subseteq R$,
$$\frac{\log(\frac{4}{|I|})^2\log(\frac{4}{|J|})^2}{|\Omega|}\sum_{Q\in\mathcal{R},Q\subseteq\Omega}|\varphi_Q|^2 \leq C.$$

3.3.1 The main paraproduct

Let $\varphi \in L^2(\mathbb{T}^2)$. The paraproduct $\pi_{\pi_\varphi} = \pi_\varphi^{(1,2)}$ is defined by
$$\pi_\varphi^{(1,2)}f = \sum_{i,j\geq 0}(\Delta_{i,j}\varphi)(E_{i,j}f) = \sum_{R\in\mathcal{R}}h_R\varphi_R m_R f$$
on functions with finite Haar expansion. We will sometimes write π_φ or $\pi[\varphi]$ to avoid ambiguities with complicated symbols. In this section, we show that π_φ extends as a bounded operator on $BMO^d(\mathbb{T}^2)$ if and only if $\varphi \in LMO^d(\mathbb{T}^2)$. For this, we first provide the reader with some useful lemmas.

Lemma 3.3.4.
$$|m_R b| \lesssim kn\|b\|_{\mathrm{BMO}^d(\mathbb{T}^2)} \quad (R\in\mathcal{R}_{n,k});$$
$$\|\chi_R b\|_2^2 \lesssim k^2n^2|R|\|b\|_{\mathrm{BMO}^d(\mathbb{T}^2)}^2 \quad (R\in\mathcal{R}_{k,n});$$
$$\|m_I b\|_{\mathrm{BMO}^d(\mathbb{T})} \lesssim k\|b\|_{\mathrm{BMO}^d(\mathbb{T}^2)} \quad (I\in\mathcal{D}_k);$$
and this is sharp.

Proof. Let $R = I \times J$. For the first inequality, consider
$$\begin{aligned}
\sup_{b\in\mathrm{BMO}^d,\|b\|_{\mathrm{BMO}^d}=1}|m_R b| &= \sup_{b\in\mathrm{BMO}^d,\|b\|_{\mathrm{BMO}^d}=1}|\langle b,\frac{\chi_R}{|R|}\rangle| \\
&\lesssim \|\frac{\chi_R}{|R|}\|_{H_d^1(\mathbb{T}^2)} \\
&= \|\frac{\chi_I}{|J|}\|_{H_d^1(\mathbb{T})}\|\frac{\chi_J}{|J|}\|_{H_d^1(\mathbb{T})} \\
&\lesssim \log(\frac{4}{|I|})\log(\frac{4}{|J|}) \approx kn,
\end{aligned}$$

where we use the $H^1_d(\mathbb{T}^2)$- $\mathrm{BMO}^d(\mathbb{T}^2)$ duality in the first line and the known one-variable results in the last line.

For the second inequality, note that $\chi_R b(s,t) = P_R b(s,t) + \chi_R(s,t) m_I b(t) + \chi_R(s,t) m_J b(s) - \chi_R(s,t) m_R b$ (see [21]).

Clearly $\|P_R b\|_2^2 \leq |R| \|b\|^2_{\mathrm{BMO}^d}$ and $\|\chi_R m_R b\|_2^2 = |m_R b|^2 |R| \lesssim n^2 k^2 |R| \|b\|^2_{\mathrm{BMO}^d}$ by the first inequality. The results for the remaining terms follow from the one-dimensional John-Nirenberg inequality, since e. g.

$$\begin{aligned}
\|\chi_R(s,t) m_I b(t)\|_2 &= \sup_{f \in L^2(\mathbb{T}), \|f\|_2 \leq 1} |I|^{1/2} \left| \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{1}{|I|} \chi_R(s,t) f(t) b(s,t) ds dt \right| \\
&\leq |I|^{1/2} \|b\|_{\mathrm{BMO}^d} \sup_{f \in L^2(\mathbb{T}), \|f\|_2 \leq 1} \left\| \frac{1}{|I|} \chi_R(s,t) f(t) \right\|_{H^1_d(\mathbb{T}^2)} \\
&= |I|^{1/2} \|b\|_{\mathrm{BMO}^d} \sup_{f \in L^2(\mathbb{T}), \|f\|_2 \leq 1} \left\| \frac{1}{|I|} \chi_I(s) \right\|_{H^1_d(\mathbb{T})} \|\chi_J(t) f(t)\|_{H^1_d(\mathbb{T})} \\
&\lesssim k |I|^{1/2} \|b\|_{\mathrm{BMO}^d} \sup_{\psi \in \mathrm{BMO}^d(\mathbb{T})} \|\chi_J \psi\|_2 \lesssim kn |I|^{1/2} |J|^{1/2} \|b\|_{\mathrm{BMO}^d}.
\end{aligned}$$

The last inequality follows in a very similar way:

$$\begin{aligned}
\|m_I b\|_{\mathrm{BMO}(\mathbb{T})} &\approx \sup_{f \in H^1_d(\mathbb{T}), \|f\|_{H^1_d} \leq 1} \left| \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{1}{|I|} \chi_I(s) f(t) b(s,t) ds dt \right| \\
&\lesssim \|b\|_{\mathrm{BMO}^d(\mathbb{T}^2)} \sup_{f \in H^1_d(\mathbb{T}), \|f\|_{H^1_d} \leq 1} \left\| \frac{1}{|I|} \chi_I(s) f(t) \right\|_{H^1_d(\mathbb{T}^2)} \\
&= \|b\|_{\mathrm{BMO}^d(\mathbb{T}^2)} \left\| \frac{1}{|I|} \chi_I(s) \right\|_{H^1_d(\mathbb{T})} \lesssim k \|b\|_{\mathrm{BMO}^d(\mathbb{T}^2)}.
\end{aligned}$$

For the sharpness in the two first inequalities, it suffices to test with $b(s,t) = f^I(s) f^J(t)$ when $R = I \times J$ and f^I is the $BMO^d(\mathbb{T})$-function given by (3.2.6). It follows easily that

$$|m_R b| \gtrsim \log \frac{4}{|I|} \log \frac{4}{|J|} \gtrsim kn;$$

and

$$||\chi_R b||_2^2 \gtrsim k^2 n^2 ||\chi_R||_2^2 = k^2 n^2 |R|.$$

For the last inequality, we take for I fixed, $b(s,t) = f^I(s) g(t)$ where f^I is the $BMO^d(\mathbb{T})$-function given by (3.2.6) and $g \in BMO^d(\mathbb{T})$ with $||g||_{BMO^d(\mathbb{T})} = 1$. Recalling that $f^I \gtrsim k$ on I, we obtain easily that

$$||m_I b||_{BMO^d(\mathbb{T})} \gtrsim k ||g||_{BMO^d(\mathbb{T})} = k.$$

$\square$

We also have the following useful lemma, in the same vein, which will be also of use in the next chapter.

Lemma 3.3.5. *For any open set $\mathcal{U} \subset \mathbb{T}$,*

$$\|\chi_I P_{\mathcal{U}} b\|_2^2 \lesssim k^2 |I||\mathcal{U}|\|b\|^2_{\mathrm{BMO}^d(\mathbb{T}^2)} \quad (I \in \mathcal{D}_k).$$

Proof. Let us first remark that any open subset of $\mathbb{T}$ can be written as a disjoint union of countably many dyadic intervals. Let write $\mathcal{U}$ as the countable union of its maximal dyadic subintervals. Thus $P_{\mathcal{U}}$ is the sum of the mutually orthogonal projections corresponding to these maximal subintervals. Consequently, we only need to prove the lemma for the case where $\Omega = J$ is a dyadic interval. For this, we recall that $\chi_I P_J b = P_{I\times J} - \chi_I m_I(P_J b)$ and that $\|P_{I\times J} b\|_2^2 \leq |I||J|\|b\|^2_{\mathrm{BMO}^d}$. Thus, we only have to estimate the second term. Using the one dimensional version of the first inequality in the previous lemma we obtain

$$\begin{aligned}
\|\chi_I(t) m_I(P_J b)(s)\|_2^2 &\lesssim |I|\|P_J(m_I b)\|^2_{L^2(\mathbb{T})} \\
&\lesssim |I|k^2 \|\|P_J b\|_{BMO^d(\mathbb{T})}\|^2_{L^2(\mathbb{T})} \\
&\lesssim |I||J|\|m_I b\|^2_{\mathrm{BMO}^d(\mathbb{T})} \\
&\lesssim |I||J|k^2 \|b\|^2_{\mathrm{BMO}^d}.
\end{aligned}$$

□

The next lemma provides an important identity for our study.

Lemma 3.3.6. *Let $b \in L^2(\mathbb{T}^2)$ and let $k, l \in \mathbb{N}$. Then*

$$\|\pi_b E_{k,l}\|_{L^2\to L^2} = \|\pi_{\tilde{b}}\|$$

where

$$\tilde{b}_{I,J} = \begin{cases} b_{I,J} & \text{if} \quad |I| > 2^{-k}, |J| > 2^{-l} \\ (\sum_{J'\subseteq J} |b_{I,J'}|^2)^{1/2} & \text{if} \quad |I| > 2^{-k}, |J| = 2^{-l} \\ (\sum_{I'\subseteq I} |b_{I',J}|^2)^{1/2} & \text{if} \quad |I| = 2^{-k}, |J| > 2^{-l} \\ (\sum_{I'\subseteq I, J'\subseteq J} |b_{I',J'}|^2)^{1/2} & \text{if} \quad |I| = 2^{-k}, |J| = 2^{-l} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $f \in L^2(\mathbb{T}^2)$. Then

$$\begin{aligned}
\|\pi_b E_{k,l} f\|_2^2 &= \|\sum_{i,j} (\Delta_{i,j} b) E_{i,j} E_{k,l} f\|^2 \\
&= \sum_{i\geq k, j\geq l} \|(\Delta_{i,j} b) E_{k,l} f\|^2 + \sum_{i\geq k, j<l} \|(\Delta_{i,j} b) E_j^{(2)} E_{k,l} f\|^2 \\
&\quad + \sum_{i<k, j\geq l} \|(\Delta_{i,j} b) E_i^{(1)} E_{k,l} f\|^2 + \sum_{i<k, j<l} \|(\Delta_{i,j} b) E_{i,j} E_{k,l} f\|^2 \\
&= \|(\sum_{i\geq k, j\geq l} |\Delta_{i,j} b|^2)^{1/2} E_{k,l} f\|_2^2 + \sum_{j<l} \|(\sum_{i\geq k} |\Delta_{i,j} b|^2)^{1/2} E_{k,j} f\|^2 \\
&\quad + \sum_{i<k} \|(\sum_{j\geq l} |\Delta_{i,j} b|^2)^{1/2} E_{i,l} f\|^2 + \sum_{i<k, j<l} \|(\Delta_{i,j} b) E_{i,j} f\|^2 \\
&= \|\sum_{i\leq k, j\leq l} (\Delta_{i,j} \tilde{b}) E_{i,j} f\|^2 = \|\pi_{\tilde{b}} f\|^2.
\end{aligned}$$

$\square$

Here is our main technical lemma.

Lemma 3.3.7. *Let* $\varphi \in \mathrm{LMO}^d(\mathbb{T}^2)$ *and* $b \in \mathrm{BMO}^d(\mathbb{T}^2)$. *Then*

$$\|\pi\left[\pi[Q_{i,j}\varphi] b\right] E_{k,l}\|_{L^2\to L^2} = \|\pi_{\left[\pi_{Q_{i,j}\varphi} b\right]} E_{k,l}\|_{L^2\to L^2} \lesssim \frac{kl}{ij} \|\varphi\|_{\mathrm{LMO}^d} \|b\|_{\mathrm{BMO}^d}.$$

Proof. We can assume that $\varphi = Q_{ij}\varphi$. By Lemma 3.3.6, we have to estimate the BMO^d norm of $\widetilde{\pi_{Q_{i,j}\varphi} b}$. Clearly

$$\begin{aligned}
\widetilde{\pi_\varphi b} &= \widetilde{E_{k,l}\pi_\varphi b} + E_k^{(1)} \widetilde{Q_l^{(2)}} \pi_\varphi b + E_l^{(2)} \widetilde{Q_k^{(1)}} \pi_\varphi b + \widetilde{Q_k^{(1)} Q_l^{(2)}} \pi_\varphi b \\
&= E_{k,l}\pi_\varphi b + E_k^{(1)} \widetilde{Q_l^{(2)} \pi_\varphi b} + E_l^{(2)} \widetilde{Q_k^{(1)} \pi_\varphi b} + Q_k^{(1)} \widetilde{Q_l^{(2)} \pi_\varphi b} = I + II + III + IV.
\end{aligned}$$

(indeed, this is the decomposition given by formula (3.2.4) and the definition of $\tilde{b}$ in Lemma 3.3.6).

We start with term I. For any open set $\Omega \subseteq \mathbb{T}^2$,

$$\begin{aligned}
\frac{1}{|\Omega|} \|P_\Omega E_{k,l} \pi_\varphi b\|_2 &= \frac{1}{|\Omega|} \sum_{R=I\times J, |I|>2^{-k}, |J|>2^{-l}, R\subset\Omega} |\varphi_R|^2 |m_R b|^2 \\
&\lesssim \frac{k^2 l^2}{|\Omega|} \sum_{R=I\times J, |I|>2^{-k}, |J|>2^{-l}, R\subset\Omega} |\varphi_R|^2 \|b\|_{\mathrm{BMO}^d}^2 \\
&\lesssim k^2 l^2 \|\varphi\|_{\mathrm{BMO}^d}^2 \|b\|_{\mathrm{BMO}^d}^2 \lesssim \frac{k^2 l^2}{i^2 j^2} \|\varphi\|_{\mathrm{LMO}^d}^2 \|b\|_{\mathrm{BMO}^d}^2.
\end{aligned}$$

by the definition of LMO^d and by Lemma 3.3.4.

For term II, note that since $E_k^{(1)} \widetilde{Q_l^{(2)} \pi_\varphi b}$ has only nontrivial Haar coefficients for those $R = I \times J$ with $|J| = 2^{-l}$ and $|I| > 2^{-k}$ (note that this corresponds to the second term in the definition of $\tilde{b}$ in Lemma 3.3.6), it is sufficient to check the BMO norm on rectangles $R = I \times J$ with $|J| = 2^{-l}$ and $|I| > 2^{-k}$. Then

$$\begin{aligned}\frac{1}{|R|}\|P_R(E_k^{(1)}\widetilde{Q_l^{(2)}}\pi_\varphi b)\|_2^2 &= \frac{1}{|R|}\sum_{I'\subseteq I}|(E_k^{(1)}\widetilde{Q_l^{(2)}}\pi_\varphi b)_{I',J}|^2\\ &= \frac{1}{|R|}\sum_{I'\subseteq I, J'\subseteq J, I'\times J'\in\mathcal{R}}|\varphi_{I'\times J'}|^2|m_{I'\times J'}b|^2\\ &= \frac{1}{|R|}\|\pi_\varphi\chi_R b\|_2^2 \lesssim \|\varphi\|_{\mathrm{BMO}^d}^2\frac{1}{|R|}\|\chi_R b\|_2^2 \lesssim \frac{k^2l^2}{i^2j^2}\|\varphi\|_{\mathrm{LMO}^d}^2\|b\|_{\mathrm{BMO}^d}^2.\end{aligned}$$

Term III is dealt with analogously. For term IV, note that since $Q_k^{(1)}\widetilde{Q_l^{(2)}}\pi_\varphi b$ has only nontrivial Haar coefficient for $R \in \mathcal{R}_{k,l}$, it is enough to check the BMO norm on rectangles of this type, and we obtain for $R = I \times J \in \mathcal{R}_{k,l}$:

$$\begin{aligned}\frac{1}{|R|}\int_R|P_R(Q_k^{(1)}\widetilde{Q_l^{(2)}}\pi_\varphi b)|^2 ds dt &= \frac{1}{|I||J|}\sum_{I'\subseteq I, J'\subseteq J}|\varphi_{I',J'}|^2|m_{I',J'}b|^2\\ &= \frac{1}{|R|}\|\pi_\varphi\chi_R b\|_2^2\\ &\lesssim \frac{1}{|R|}\|\varphi\|_{\mathrm{BMO}^d}^2\|\chi_R b\|_2^2\\ &\lesssim \frac{k^2l^2}{i^2j^2}\|\varphi\|_{\mathrm{LMO}^d}^2\|b\|_{\mathrm{BMO}^d}^2\end{aligned}$$

by Lemma 3.3.4. □

Let us now recall the following Cotlar's lemma (see [101]).

Lemma 3.3.8. *Suppose $\{T_j\}$ is a finite collection of bounded operators on L^2. We denote the adjoint T_j by T_j^*. We assume that we are given a sequence of positive constants $\{\alpha(j)\}_{j=-\infty}^{\infty}$, with*

$$A = \sum_{j=-\infty}^{\infty}\alpha(j) < \infty,$$

and

$$||T_i^*T_j||_{L^2\to L^2} \leq \big[\alpha(i-j)\big]^2,$$

$$||T_iT_j^*||_{L^2\to L^2} \leq \big[\alpha(i-j)\big]^2.$$

Then the operator

$$T = \sum_j T_j$$

satisfies

$$||T||_{L^2\to L^2} \leq A.$$

Theorem 3.3.9. *Let $\varphi \in L^2(\mathbb{T}^2)$. Then $\varphi \in \mathrm{LMO}^d(\mathbb{T}^2)$, if and only if $\pi_\varphi : \mathrm{BMO}^d(\mathbb{T}^2) \to \mathrm{BMO}^d(\mathbb{T}^2)$ is bounded, and $\|\pi_\varphi\|_{\mathrm{BMO}^d \to \mathrm{BMO}^d} \approx \|\varphi\|_{\mathrm{LMO}^d}$.*

Proof. We begin by proving necessity. Suppose that $\pi_\varphi : \mathrm{BMO}^d(\mathbb{T}^2) \to \mathrm{BMO}^d(\mathbb{T}^2)$ is bounded. Let $R = I \times J$ be a dyadic rectangle, with $|I| = 2^{-k}$ and $|J| = 2^{-l}$, and let $\Omega \subseteq R$ be open. From Lemma 3.2.5, it is easy to see that there exists a function $b \in \mathrm{BMO}^d$ with $b|_R \equiv kl$ and $\|b\|_{\mathrm{BMO}^d} \leq C$, where C is a constant independent of R. Such a function is obtained as a tensor product of two one-variable functions b_1, b_2 in the variables s, t respectively, which have the corresponding properties for the intervals I and J, respectively and are given by (3.2.6). Then

$$\frac{\left(\log(\frac{4}{|I|})\right)^2 \left(\log(\frac{4}{|J|})\right)^2}{|\Omega|} \sum_{Q\in\mathcal{R}, Q\subseteq\Omega} |\varphi_Q|^2 \approx \frac{kl}{|\Omega|} \sum_{Q\in\mathcal{R}, Q\subseteq\Omega} |\varphi_Q|^2$$
$$= \frac{1}{|\Omega|} \sum_{Q\in\mathcal{R}, Q\subseteq\Omega} |\varphi_Q|^2 |m_Q b|^2 \leq \|\pi_\varphi b\|^2_{\mathrm{BMO}^d} \leq C^2 \|\pi_\varphi\|^2_{\mathrm{BMO}^d \to \mathrm{BMO}^d}.$$

Thus $\varphi \in \mathrm{LMO}^d(\mathbb{T}^2)$ by Remark 3.3.3, with the appropriate norm estimate.

To prove sufficiency of the $\mathrm{LMO}^d(\mathbb{T}^2)$ condition for boundedness of the paraproduct on $\mathrm{BMO}^d(\mathbb{T}^2)$, let $\varphi \in \mathrm{LMO}^d(\mathbb{T}^2)$ and $b \in \mathrm{BMO}^d(\mathbb{T}^2)$. Assume that b has a finite Haar expansion. We will estimate $\|\pi_\varphi b\|_{\mathrm{BMO}^d} \approx \|\pi[\pi_\varphi b]\|_{L^2 \to L^2}$ by means of Cotlar's Lemma.

For $N, K \in \mathbb{N}$, let

$$\begin{aligned} P_{N,K} &= \sum_{i=2^N}^{2^{N+1}-1} \sum_{j=2^K}^{2^{K+1}-1} \Delta_{i,j}, \\ P^{N,K} &= \sum_{i=2^N}^{\infty} \sum_{j=2^K}^{\infty} \Delta_{i,j}, \end{aligned} \tag{3.3.1}$$

and

$$T_{N,K} = \pi[\pi_\varphi b] P_{N,K}.$$

That means, we wish to estimate the $L^2 - L^2$ operator norm of $\pi[\pi_\varphi b] = \sum_{N,K=0}^{\infty} T_{N,K}$. In fact since we suppose that b has a finite Haar expansion, and in the aim of applying Cotlar's Lemma, we only need to consider a finite family of the operators $\{T_{N,K}\}_{N,K\geq 0}$. The result for the paraproduct $\pi[\pi_\varphi b]$ will follow by taking the limits.

Clearly $T_{N,K} T^*_{N',K'} = 0$ for $N \neq N'$ or $K \neq K'$. Therefore, we only have to estimate the norm of $T^*_{N,K} T_{N',K'}$ for $N, N', K, K' \in \mathbb{N}$. Letting $\overline{N} = \max\{N, N'\}$, $\underline{N} = \min\{N, N'\}$,

$\overline{K} = \max\{K, K'\}$, $\underline{K} = \min\{K, K'\}$, we obtain

$$\begin{aligned}
\|T^*_{N,K} T_{N',K'}\| &= \|P_{N,K} \left(\pi[\pi_\varphi b]\right)^* \left(\pi[\pi_\varphi b]\right) P_{N',K'}\| \\
&= \|P_{N,K} \left(\pi[P^{N,K}\pi_\varphi b]\right)^* \left(\pi[P^{N',K'}\pi_\varphi b]\right) P_{N',K'}\| \\
&= \|P_{N,K} \left(\pi[P^{\overline{N},\overline{K}}\pi_\varphi b]\right)^* \left(\pi[P^{\overline{N},\overline{K}}\pi_\varphi b]\right) P_{N',K'}\| \\
&\le \|\pi[P^{\overline{N},\overline{K}}\pi_\varphi b] P_{N,K}\| \|\pi[P^{\overline{N},\overline{K}}\pi_\varphi b] P_{N',K'}\| \\
&= \|\pi[\pi[P^{\overline{N},\overline{K}}\varphi] b] P_{N,K}\| \|\pi[\pi[P^{\overline{N},\overline{K}}\varphi] b] P_{N',K'}\| \\
&\lesssim \frac{2^{\underline{N}+1} 2^{\underline{K}+1}}{2^{\overline{N}} 2^{\overline{K}}} \frac{2^{\overline{N}+1} 2^{\overline{K}+1}}{2^{\overline{N}} 2^{\overline{K}}} \|\varphi\|^2_{\mathrm{LMO}^d} \|b\|^2_{\mathrm{BMO}^d} \\
&\lesssim 2^{-|N-N'|} 2^{-|K-K'|} \|\varphi\|^2_{\mathrm{LMO}^d} \|b\|^2_{\mathrm{BMO}^d}
\end{aligned}$$

by Lemma 3.3.7. Thus, by Cotlar's Lemma, $T = \pi[\pi_\varphi b]$ is bounded on $L^2(\mathbb{T}^2)$, and there exists an absolute constant $C > 0$ with

$$\|\pi[\pi_\varphi b]\| \le C \|\varphi\|_{\mathrm{LMO}^d} \|b\|_{\mathrm{BMO}^d}.$$

Consequently,

$$\|\pi_\varphi b\|_{\mathrm{BMO}^d} \lesssim \|\varphi\|_{\mathrm{LMO}^d} \|b\|_{\mathrm{BMO}^d}.$$

□

3.3.2 The other paraproducts

There are four "good" dyadic paraproducts in two variables, namely the paraproduct π discussed above, its adjoint defined by

$$\Delta_\varphi f = \Delta_{\Delta_\varphi} f = \sum_{R \in \mathcal{R}} \frac{\chi_R}{|R|} \varphi_R f_R,$$

and the mixed paraproducts Δ_π and π_Δ, given by

$$\Delta_\pi[\varphi] f = \Delta_{\pi_\varphi} f = \sum_{I \times J \in \mathcal{R}} \frac{\chi_I(s)}{|I|} h_J(t) \varphi_{I \times J} m_J f_I,$$

$$\pi_\Delta[\varphi] f = \Pi_{\Delta_\varphi} f = \sum_{I \times J \in \mathcal{R}} h_I(s) \frac{\chi_J(t)}{|J|} \varphi_{I \times J} m_I f_J,$$

see [21].

Interestingly, all four paraproducts have a different boundedness behaviour on $\mathrm{BMO}^d(\mathbb{T}^2)$.

Theorem 3.3.10. *Let $\varphi \in L^2(\mathbb{T}^2)$. Then*

(1) $\Delta_\varphi : \mathrm{BMO}^d(\mathbb{T}^2) \to \mathrm{BMO}^d(\mathbb{T}^2)$ is bounded, if and only if $\varphi \in \mathrm{BMO}^d$.

Moreover, $\|\Delta_\varphi\|_{\mathrm{BMO}^d \to \mathrm{BMO}^d} \approx \|\varphi\|_{\mathrm{BMO}^d}$.

(2) $\pi_\Delta[\varphi] : \mathrm{BMO}^d(\mathbb{T}^2) \to \mathrm{BMO}^d(\mathbb{T}^2)$ *is bounded, if* $\varphi \in \mathrm{LMO}_1^d(\mathbb{T}^2)$.
Moreover, $\|\pi_\Delta[\varphi]\|_{\mathrm{BMO}^d(\mathbb{T}^2)\to\mathrm{BMO}^d(\mathbb{T}^2)} \lesssim \|\varphi\|_{\mathrm{LMO}_1^d(\mathbb{T}^2)}$.

(3) $\Delta_\pi[\varphi] : \mathrm{BMO}^d(\mathbb{T}^2) \to \mathrm{BMO}^d(\mathbb{T}^2)$ *is bounded, if* $\varphi \in \mathrm{LMO}_2^d(\mathbb{T}^2)$.
Moreover, $\|\Delta_\pi[\varphi]\|_{\mathrm{BMO}^d(\mathbb{T}^2)\to\mathrm{BMO}^d(\mathbb{T}^2)} \lesssim \|\varphi\|_{\mathrm{LMO}_2^d(\mathbb{T}^2)}$.

Proof. (1) was shown in [21]. To show (2), we will follow a simplified version of the ideas of the proof of Theorem 3.3.9.

Lemma 3.3.11. *Let* $b \in L^2(\mathbb{T}^2)$ *and let* $k \in \mathbb{N}$. *Then*

$$\|\pi_b E_k^{(1)}\|_{L^2\to L^2} = \|\pi_{\tilde{b}}\|$$

where

$$\tilde{b}_{I,J} = \begin{cases} b_{I,J} & \text{if} \quad |I| > 2^{-k} \\ (\sum_{I'\subseteq I} |b_{I',J}|^2)^{1/2} & \text{if} \quad |I| = 2^{-k} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. As in Lemma 3.3.6. $\square$

Lemma 3.3.12. *Let* $\varphi \in \mathrm{LMO}_1^d(\mathbb{T}^2)$ *and* $b \in \mathrm{BMO}^d(\mathbb{T}^2)$, $i, k \in \mathbb{N}$. *Then*

$$\|\pi\left[\pi_\Delta[Q_i^{(1)}\varphi]b\right]E_k^{(1)}\|_{L^2\to L^2} \lesssim \frac{k}{i}\|\varphi\|_{\mathrm{LMO}_1^d}\|b\|_{\mathrm{BMO}^d}.$$

Proof. We write Q for $Q^{(1)}$ and E for $E^{(1)}$. We can assume that $\varphi = Q_i\varphi$. Following the results in Lemma 3.3.11, we estimate

$$\|\widetilde{\pi_{\Delta\varphi}b}\|_{\mathrm{BMO}^d} \le \|\widetilde{\pi_{\Delta E_k\varphi}b}\|_{\mathrm{BMO}^d} + \|\widetilde{\pi_{\Delta Q_k\varphi}b}\|_{\mathrm{BMO}^d}.$$

We start with the second term and remember that still $Q_iQ_k\varphi = Q_k\varphi$. Since $\pi_\Delta[Q_k\varphi]b$ has no nontrivial Haar terms in the first variable for intervals I with $|I| > 2^{-k}$,

$$\pi_\Delta\widetilde{[Q_k\varphi]}b = \sum_{J\in\mathcal{D}}\sum_{|I|=2^{-k}} h_I(s)(\sum_{I'\subseteq I}|\varphi_{I'J}|^2|m_{I'}b_J|^2)^{1/2}\frac{\chi_J}{|J|}(t),$$

and this has only nontrivial Haar terms in the first variable for intervals I with $|I| = 2^{-k}$. The computation of the BMO^d norm is therefore very easy: instead of considering general open sets, one only has to consider rectangles of the form $R = I \times J$, $|I| = 2^{-k}$. Thus, using Lemma 3.3.5 we obtain

$$\begin{aligned}
\|P_R(\pi_\Delta\widetilde{[Q_k\varphi]}b)\|_2^2 &= \|P_R(\pi_\Delta\widetilde{[Q_k\varphi]}b)\|_2^2 \\
&\leq \|\pi_\Delta\widetilde{[P_R Q_k}\varphi]b\|_2^2 \\
&= \|\pi_\Delta[P_R Q_k\varphi]b\|_2^2 \\
&= \|\sum_{J'\subseteq J}\sum_{I'\subseteq I} h_{I'}\frac{\chi_{J'}}{|J'|}(t)\varphi_{I'J'}m_{I'}b_{J'}\|_2^2 \\
&= \|\pi_{\Delta_{P_R\varphi}}\chi_I(s)P_J b\|_2^2 \\
&\lesssim \|P_R\varphi\|_{\mathrm{BMO}^d}\|\chi_I P_J b\|_2^2 \lesssim \frac{k^2}{i^2}\|\varphi\|_{\mathrm{LMO}^d}^2\|b\|_{\mathrm{BMO}^d}^2.
\end{aligned}$$

Now we have to deal with the first term $\|\widetilde{\pi_{\Delta_{E_k\varphi}}}b\|_{\mathrm{BMO}^d}$. Again, we recall that still $E_k\varphi = Q_i E_k\varphi$ with this notation. Let $\Omega \subseteq \mathbb{T}^2$ be open and write

$$\mathcal{J}_I = \cup_{J\in\mathcal{D}, I\times J\subseteq\Omega} J \ \text{ for } \ I\in\mathcal{D}. \tag{3.3.2}$$

Then

$$\begin{aligned}
\|P_\Omega(\widetilde{\pi_{\Delta_{E_k\varphi}}}b)\|_2^2 &= \|P_\Omega\left(\pi_\Delta[\widetilde{E_k\varphi}]\right)\|_2^2 \\
&= \|P_\Omega \sum_{I,J\in\mathcal{D},|I|>2^{-k}} h_I(s)\frac{\chi_J}{|J|}(t)\varphi_{IJ}m_I b_J\|_2^2 \\
&\leq \|\sum_{I\in\mathcal{D},|I|>2^{-k}}\ \sum_{J\in\mathcal{D}:I\times J\subseteq\Omega} h_I(s)\frac{\chi_J}{|J|}(t)\varphi_{IJ}m_I b_J\|_2^2 \\
&= \sum_{I\in\mathcal{D},|I|>2^{-k}} \|\sum_{J\subseteq\mathcal{J}_I}\frac{\chi_J}{|J|}(t)\varphi_{IJ}m_I b_J\|_2^2 \\
&= \sum_{I\in\mathcal{D},|I|>2^{-k}} \|\Delta_{m_I b}P_{\mathcal{J}_I}\varphi_I\|_2^2 \lesssim \sum_{I\in\mathcal{D},|I|>2^{-k}} \|m_I b\|_{\mathrm{BMO}^d}^2\|P_{\mathcal{J}_I}\varphi_I\|_2^2 \\
&\lesssim k^2\|b\|_{\mathrm{BMO}^d}^2\sum_{I\in\mathcal{D}}\|P_{\mathcal{J}_I}\varphi_I\|_2^2 \\
&\lesssim k^2\|b\|_{\mathrm{BMO}^d}^2\|P_\Omega\varphi\|_2^2 \lesssim \frac{k^2}{i^2}\|b\|_{\mathrm{BMO}^d}^2\|\varphi\|_{\mathrm{LMO}^d}^2|\Omega|
\end{aligned}$$

by Lemma 3.3.4. □

The remainder of the proof of (2) is now exactly analogous to the proof of Therem 3.3.9, defining $T_N = \pi\big[\pi_\Delta[\varphi]\big]P_N$, where $P_N = \sum_{i=2^N}^{2^{N+1}-1}\Delta_i^{(1)}$, and using Cotlar's Lemma in one parameter. Finally, (3) follows by simply switching variables. □

Chapter 4

Pointwise multipliers of product BMO

We characterize in this chapter the set of pointwise multipliers of the product space of functions of bounded mean oscillation. This work is motivated by the one dimensional result of [100] on the algebra of pointwise multipliers of this space.

4.1 Functions of bounded mean oscillation

In this section, we recall various notions of functions of bounded mean oscillation first in the torus and then in the bitorus. In the product setting, we also recall equivalent definitions of the little (small) BMO space denoted bmo and introduce the little LMO space denoted by lmo.

4.1.1 Bounded mean oscillation in one dimension

Originally, we say a function f is in the space of functions of bounded mean oscillation BMO if

$$||f||_* := \sup_I \frac{1}{|I|}\int_I |b(t) - m_I b|dt < \infty,$$

where $m_I f = \frac{1}{|I|}\int_I f(t)dt$ is the mean of the function f over the interval I. Because of the John-Nirenberg's inequality:

$f \in BMO$ if and only if there exist $C,\ c > 0$ such that

$$|\{t \in I : |f(t) - m_I f| > \lambda\}| \leq C|I|e^{-c\lambda/||f||_*},$$

we observe that the following equivalence holds

$$||f||_* \approx \left(\sup_I \frac{1}{|I|}\int_I |b(t)-m_I b|^p dt\right)^{1/p}. \tag{4.1.1}$$

Remark 4.1.1. *The equivalence (4.1.1) explains the choice of the exponent 2 in the definition of $BMO^d(\mathbb{T})$ in previous chapter.*

The Hilbert transform is defined on $L^1(\mathbb{T})$ by

$$Hf(x) := \text{p.v.}\frac{1}{\pi}\int_0^1 \frac{f(y)}{\tan(\pi(x-y))}dy. \tag{4.1.2}$$

A second characterization of BMO is in terms of duality with the real Hardy space (see [41]):

$$BMO(\mathbb{T}) = \left(H^1_{Re}(\mathbb{T})\right)^*$$

where $\left(H^1_{Re}(\mathbb{T})\right)^*$ is the dual space with respect to the (L^2, L^2) duality, of the real Hardy space $H^1_{Re}(\mathbb{T})$ defined as

$$H^1_{Re}(\mathbb{T}) = \{f \in L^1(\mathbb{T}) : Hf \in L^1(\mathbb{T})\}.$$

Let us now introduce the useful notion of function of logarithmic mean oscillation. We say a function f has logarithmic mean oscillation if

$$\|f\|_{*,log} := \sup_{I\subset\mathbb{T}} \frac{\left(\log\frac{4}{|I|}\right)^2}{|I|}\int_I |f(t)-m_I f|^2 dt < \infty. \tag{4.1.3}$$

The space of functions of logarithmic mean oscillation is denoted LMO($\mathbb{T}$). Taking only dyadic intervals in the definitions of BMO($\mathbb{T}$) and LMO($\mathbb{T}$), we obtain their dyadic counterparts already introduced in Chapter 3 BMO^d($\mathbb{T}$) and LMO^d($\mathbb{T}$) respectively. The following one dimensional version of Theorem 3.3.9 is easily obtained.

Proposition 4.1.2. *For a function φ in $L^2(\mathbb{T})$, we have the equivalence*

$$\|\pi_\varphi\|_{\mathrm{BMO}^d\to\mathrm{BMO}^d} \simeq \|\varphi\|_{*,log}.$$

4.1.2 Product space of functions of bounded mean oscillation

The product BMO($\mathbb{T}^2$) of S-Y. A. Chang and R. Fefferman [29–31] is the dual of the real product Hardy space $H^1_{Re}(\mathbb{T}^2)$ defined by

$$H^1_{Re}(\mathbb{T}^2) = \{f \in L^1 : H_1 f \in L^1, H_2 f \in L^1, H_1H_2 f \in L^1\},$$

where H_1 and H_2 are Hilbert transforms in the first and the second variable. Keeping in mind the equivalent definitions in one dimension, we also recall the definition of the small BMO space $\mathrm{bmo}(\mathbb{T}^2)$ introduced by Cotlar and Sadosky [37]. A function f belongs to the space of functions of bounded mean oscillation on rectangles, $\mathrm{bmo}(\mathbb{T}^2)$, if

$$\sup_{R\subset\mathbb{T}^2 \text{ rectangle}} \frac{1}{|R|}\int_R |b(s,t)-m_R b|dsdt < \infty.$$

In the one dimensional case, $\mathrm{bmo}(\mathbb{T}) = \mathrm{BMO}(\mathbb{T})$. In higher dimension, this is far from being the case. In fact we have $\mathrm{bmo}(\mathbb{T}^2) \subset \mathrm{BMO}(\mathbb{T}^2)$ in the strict sense (see [37, 46]). Nevertheless, equivalent definitions of product $\mathrm{BMO}(\mathbb{T}^2)$ are obtained as in one dimension in terms of range of symbols of bounded Hankel operators or commutators (see [30,45,46]).

The little BMO space on the bitorus $\mathrm{bmo}(\mathbb{T}^2)$ can also be characterised as

$$\begin{aligned}\mathrm{bmo}(\mathbb{T}^2) = \{b\in L^2(\mathbb{T}^2) : \exists C>0 \text{ such that } &\|b(\cdot,t)\|_{\mathrm{BMO}(\mathbb{T})}\le C,\\ &\|b(s,\cdot)\|_{\mathrm{BMO}(\mathbb{T})}\le C \text{ for a.e. } t,s\in\mathbb{T}\}\end{aligned} \quad (4.1.4)$$

and as

$$\begin{aligned}\mathrm{bmo}(\mathbb{T}^2) = \{b\in L^2(\mathbb{T}^2) : \exists C>0 \text{ such that } &\|m_I^{(1)}b\|_{\mathrm{BMO}(\mathbb{T})}\le C,\\ &\|m_J^{(2)}b\|_{\mathrm{BMO}(\mathbb{T})}\le C \text{ for all intervals } I,J\subset\mathbb{T}\}\end{aligned} \quad (4.1.5)$$

(see [37, 46]). Here and in the following, we think of I as an interval in the first variable and J as an interval in the second variable, meaning that $m_I^{(1)}b$ is a function in the second variable and $m_J^{(2)}b$ is a function in the first variable. In the same spirit, we introduce the little LMO space in the bitorus $\mathrm{lmo}(\mathbb{T}^2)$ as follows:

$$\begin{aligned}\mathrm{lmo}(\mathbb{T}^2) = \{b\in L^2(\mathbb{T}^2) : \exists C>0 \text{ such that } &\|m_I^{(1)}b\|_{\mathrm{LMO}(\mathbb{T})}\le C,\\ &\|m_J^{(2)}b\|_{\mathrm{LMO}(\mathbb{T})}\le C \text{ for all intervals } I,J\subset\mathbb{T}\}.\end{aligned} \quad (4.1.6)$$

Replacing rectangles by dyadic rectangles and intervals by dyadic intervals in any of the characterizations above, we obtain the dyadic $\mathrm{BMO}^d(\mathbb{T}^2)$, the small dyadic BMO space $\mathrm{bmo}^d(\mathbb{T}^2)$ and the small dyadic LMO space $\mathrm{lmo}^d(\mathbb{T}^2)$.

4.1.3 Dyadic grids and averaging

For $\alpha\in[0,1]$, let $\mathcal{D}^\alpha$ denote the translated dyadic grid on $\mathbb{T}$, that is the set of all interval of the following form

$$[\alpha+k2^{-j}, \alpha+(k+1)2^{-j}),\quad k,j\in\mathbb{Z}$$

when we identify $\mathbb{T}$ with $[0,1)$. Here $\mathcal{D}^0 = \mathcal{D}$ the standard dyadic grid on $\mathbb{T}$. For $I \in \mathcal{D}^\alpha$, we denote by h_I^α the corresponding Haar function, normalized in $L^2(\mathbb{T})$. Thus the notation $\mathrm{BMO}^{d,\alpha}$ will always means that we are defining our space of functions of bounded mean oscillation with respect to the dyadic grid $\mathcal{D}^\alpha$ or the product dyadic grid $\mathcal{R}^\alpha = \mathcal{D}^{\alpha_1} \times \mathcal{D}^{\alpha_2}$ when we are in product settings, in which case $\alpha = (\alpha_1, \alpha_2)$.

In one dimension, the theorem of Garnett and Jones [Theorem 2, [83]] relates the space BMO of functions of bounded mean oscillation to its dyadic counterpart, the dyadic BMO. Its extension to the product space is due to J. Pipher and L. A. Ward [83].

Theorem 4.1.3. *Suppose that* $\varphi^\alpha \in \mathrm{BMO}^d(\mathbb{T}^2)$ *for each* $\alpha = (\alpha_1, \alpha_2) \in [0,1] \times [0,1]$, *that* $\alpha \mapsto \varphi^\alpha$ *is measurable, and that the* $\mathrm{BMO}^d(\mathbb{T}^2)$ *norms of the functions* φ^α *are uniformly bounded, i.e there is a constant* $C_d > 0$ *such that*

$$||\varphi^\alpha||_{\mathrm{BMO}^d} \leq C_d$$

for all $\alpha \in [0,1] \times [0,1]$. *Let* $x = (x_1, x_2)$. *Suppose also that*

$$\int \varphi^\alpha(x)dx = 0 \ \textit{for all} \ \alpha \in [0,1] \times [0,1].$$

Then the translation-average

$$\varphi(x) = \int_0^1 \int_0^1 \varphi^\alpha(x+\alpha)d\alpha$$

is in $\mathrm{BMO}(\mathbb{T}^2)$, *where we identify* $\mathbb{T}^2$ *with* $[0,1) \times [0,1)$.

It is a duality consequence of the inclusion $H_d^1(\mathbb{T}^2) \subset H_{Re}^1(\mathbb{T}^2)$ (see [104]) that $\mathrm{BMO}(\mathbb{T}^2) \subset \mathrm{BMO}^{d,\alpha}(\mathbb{T}^2)$ for all $\alpha = (\alpha_1, \alpha_2) \in [0,1] \times [0,1]$. Adding this fact to the averaging results of Theorem 4.1.3, we obtain the following identification.

Corollary 4.1.4. $\mathrm{BMO}(\mathbb{T}^2) = \bigcap_{\alpha \in [0,1]\times[0,1]} \mathrm{BMO}^{d,\alpha}(\mathbb{T}^2)$.

Following the above identification of product BMO space as the intersection of all dyadic BMO spaces obtained from translated dyadic grids, we introduce the following notion of the product space of functions of logarithmic mean oscillation.

Definition 4.1.5. The space of functions of logarithmic mean oscillation on $\mathbb{T}^2$, $\mathrm{LMO}(\mathbb{T}^2)$, is the intersection of all spaces of functions of dyadic logarithmic mean oscillation $\mathrm{LMO}^{d,\alpha}(\mathbb{T}^2)$.

$$\mathrm{LMO}(\mathbb{T}^2) = \bigcap_\alpha \mathrm{LMO}^{d,\alpha}(\mathbb{T}^2) \tag{4.1.7}$$

where $\alpha = (\alpha_1, \alpha_2) \in [0,1] \times [0,1]$. Here $LMO^{d,\alpha}(\mathbb{T}^2)$ is defined as in the previous chapter taking rectangles in the product dyadic grid $\mathcal{R}^\alpha = \mathcal{D}^{\alpha_1} \times \mathcal{D}^{\alpha_1}$.

Remark 4.1.6. We also have the identification $\mathrm{lmo}(\mathbb{T}^2) = \bigcap_\alpha \mathrm{lmo}^{d,\alpha}(\mathbb{T}^2)$, $\alpha = (\alpha_1, \alpha_2) \in [0,1] \times [0,1]$. Indeed this is a direct consequence of the one dimensional identification

$$\mathrm{LMO}(\mathbb{T}) = \bigcap_{\alpha\in[0,1]} \mathrm{LMO}^{d,\alpha}(\mathbb{T}).$$

Proof. It is clear that $LMO(\mathbb{T}) \subset \bigcap_{\alpha\in[0,1]} LMO^{d,\alpha}(\mathbb{T})$. Now, let $f \in \bigcap_{\alpha\in[0,1]} LMO^{d,\alpha}(\mathbb{T})$ and suppose that there exists a constant $M > 0$ such that for any dyadic interval $J \subset \mathbb{T}$ in any dyadic grid $\mathcal{D}^\alpha$ ($\alpha \in [0,1]$),

$$\frac{\left(\log \frac{4}{|J|}\right)^2}{|J|} \int_J |f(t) - m_J f|^2 dt < M.$$

Let I be an interval in $\mathbb{T}$. Then I can be covered by a dyadic interval J in a suitable grid with length $|J| \leq 2|I|$. It follows that

$$\begin{aligned}
\frac{\left(\log \frac{4}{|I|}\right)^2}{|I|} \int_I |f(t) - m_I f|^2 dt &\leq 8\frac{\left(\log \frac{4}{|J|}\right)^2}{|J|} \int_J |f(t) - m_I f|^2 dt \\
&\leq 16\left(\log \frac{4}{|J|}\right)^2 \left(\frac{1}{|J|}\int_J |f(t) - m_J f|^2 dt + |m_J f - m_I f|^2\right) \\
&\leq 80\frac{\left(\log \frac{4}{|J|}\right)^2}{|J|} \int_J |f(t) - m_J f|^2 dt < 80M.
\end{aligned}$$

This proves that f belongs to $\mathrm{LMO}(\mathbb{T})$. $\square$

4.2 Pointwise multipliers of BMO and of its dyadic counterpart

We are interested in this section to those functions $g \in L^2(\mathbb{T}^2)$ such that the pointwise product $f \cdot g$ belongs to $\mathrm{BMO}(\mathbb{T}^2)$ for all $f \in \mathrm{BMO}(\mathbb{T}^2)$. We use here the notations introduced in Chapter 2 for the multipliers spaces: $\mathcal{M}(X,Y)$ and $\mathcal{M}(X)$ (when $X = Y$). We recall that in one dimensional case, we have $\mathcal{M}(\mathrm{BMO}(\mathbb{T})) = L^\infty(\mathbb{T}) \cap \mathrm{LMO}(\mathbb{T})$.

Theorem 4.2.1. *Let $\varphi \in \mathcal{M}(\mathrm{BMO}(\mathbb{T}^2), \mathrm{BMO}^{d,\alpha}(\mathbb{T}^2))$, $\alpha \in [0,1] \times [0,1]$. Then $\varphi \in \mathrm{lmo}^{d,\alpha}(\mathbb{T}^2) \cap \mathrm{LMO}^{d,\alpha}(\mathbb{T}^2) \cap L^\infty(\mathbb{T}^2)$.*

Proof. For simplcity, we restrict ourself to the case $\alpha = (0,0)$. To show that $\varphi \in L^\infty$, note that for each dyadic rectangle R, the function χ_R belongs to $\mathrm{BMO}^d(\mathbb{T}^2)$ and consequently $\varphi\chi_R \in \mathrm{BMO}^d(\mathbb{T}^2)$. It follows from the the definition of $\mathrm{BMO}^d(\mathbb{T}^2)$ that we have

$$\|\varphi\|_{\mathcal{M}(\mathrm{BMO}(\mathbb{T}^2),\mathrm{BMO}^d(\mathbb{T}^2))} \geq \|\varphi\chi_R\|_{\mathrm{BMO}^d} \geq \frac{1}{|\tilde{R}|^{1/2}}|\langle\varphi\chi_R, h_{\tilde{R}}\rangle| = \frac{1}{4}|m_R\varphi|.$$

where $\tilde{R}$ is the parent rectangle (with respect to both dimensions) of R. Thus $\|\varphi\|_\infty \leq 4\|\varphi\|_{\mathcal{M}(\mathrm{BMO}(\mathbb{T}^2),\mathrm{BMO}^d(\mathbb{T}^2))}$.

Given a dyadic rectangle $R = I \times J$ with $|I| = 2^{-n}$ and $|J| = 2^{-k}$, we already know that we can construct a "dyadic logarithm" ℓ with the property that $\ell \in \mathrm{BMO}(\mathbb{T}^2)$, $\ell|_R \equiv nk$ and $\|\ell\|_{\mathrm{BMO}} \leq C$, where C is an absolute constant independent of R.

For any open set $\Omega \subset R$, we obtain

$$\|P_\Omega(\varphi\ell)\|_2^2 = n^2k^2\|P_\Omega\varphi\|_2^2.$$

It follows that for each rectangle $R = I \times J \in \mathcal{R}$ and for all open set $\Omega \subset R$,

$$\frac{\left(\log\frac{4}{|I|}\right)^2\left(\log\frac{4}{|J|}\right)^2}{|\Omega|}\|P_\Omega\varphi\|_2^2 \leq \|\varphi\ell\|_{BMO^d}^2.$$

Consequently, using the equivalent definition of $LMO^d(\mathbb{T}^2)$ in Proposition 3.3.3, we conclude that $\|\varphi\|_{\mathrm{LMO}^d}^2 \leq C\|\varphi\|_{\mathcal{M}(\mathrm{BMO},\mathrm{BMO}^d)}^2$.

Now, suppose that φ has a finite Haar expansion. As in the previous chapter we break up the multiplication operator $M_\varphi : \mathrm{BMO}(\mathbb{T}^2) \to \mathrm{BMO}^d(\mathbb{T}^2)$ into the 9 dyadic operators as in Chapter 3:

$$M_\varphi = \pi_\varphi + \Delta_\varphi + \pi_{\Delta_\varphi} + \Delta_{\pi_\varphi} + R_{\Delta_\varphi} + \Delta_{R_\varphi} + R_{\pi_\varphi} + \pi_{R_\varphi} + R_{R_\varphi}$$

The first 4 terms are the dyadic paraproducts and it follows from the results of Chapter 3 that they are bounded $\mathrm{BMO}(\mathbb{T}^2) \to \mathrm{BMO}^d(\mathbb{T}^2)$ since $\varphi \in \mathrm{LMO}^d(\mathbb{T}^2)$. The remaining five operators are defined as follows.

$$R_{\Delta_\varphi}b(s,t) = \sum_{I,J} m_I(\varphi_J)b_{I,J}h_I(s)h_J^2(t),$$

$$\Delta_{R_\varphi}b(s,t) = \sum_{I,J} m_J(\varphi_I)b_{I,J}h_I^2(s)h_J(t),$$

$$\pi_{R_\varphi}b(s,t) = \sum_{I,J} m_J(\varphi_I)m_I(b_J)h_I(s)h_J(t),$$

$$R_{\pi_\varphi}b(s,t) = \sum_{I,J} m_I(\varphi_J)m_J(b_I)h_I(s)h_J(t),$$

and

$$R_{R_\varphi}b(s,t) = \sum_{I,J} b_{IJ}m_{IJ}(\varphi)h_I(s)h_J(t).$$

The operator R_R is clearly bounded on $\mathrm{BMO}^d(\mathbb{T}^2)$, since $\varphi \in L^\infty(\mathbb{T}^2)$. For term 5 and 6, note that

Lemma 4.2.2. *Let $\varphi \in L^\infty(\mathbb{T}^2)$. Then R_{Δ_φ}, Δ_{R_φ} are bounded on* $\mathrm{BMO}^d(\mathbb{T}^2)$ *and in particular bounded from* $\mathrm{BMO}(\mathbb{T}^2)$ *to* $\mathrm{BMO}^d(\mathbb{T}^2)$.

Proof. Because of the symmetry of variables, we only prove the lemma for the operator Δ_{R_φ}. We first remark that for $\varphi \in L^\infty(\mathbb{T}^2)$ we have for any interval I, $\|m_I\varphi\|_{BMO^d(\mathbb{T})} \leq 2\|\varphi\|_\infty$.

Recall that

$$\Delta_{R_\varphi} b(t,s) = \sum_{I,J} m_J(\varphi_I) b_{I,J} h_I^2(t) h_J(s).$$

For any open set $\Omega \in \mathbb{T}^2$, define the subset $\mathcal{J}_J$ for $J \in \mathcal{D}$ as in (3.3.2). For any $b \in \mathrm{BMO}^d(\mathbb{T}^2)$, we obtain

$$\begin{aligned}
\|P_\Omega \Delta_{R_\varphi} b(s,t)\|_2^2 &= \|P_\Omega \sum_{I,J} m_J(\varphi_I) b_{I,J} h_I^2(t) h_J(s)\|_2^2 \\
&= \|P_\Omega \left(\sum_{J\in\mathcal{D}} (\Delta_{m_J\varphi} b_J)\, h_J(s) \right) \|_2^2 \\
&= \|P_\Omega \left(\sum_{J\in\mathcal{D}} (\Delta_{m_J\varphi} P_{\mathcal{J}_J} b_J)\, h_J(s) \right) \|_2^2 \\
&\leq \sum_{J\in\mathcal{D}} \|\Delta_{m_J\varphi}(P_{\mathcal{J}_J} b_J)\|_{L^2(\mathbb{T})}^2 \\
&\leq \sum_{J\in\mathcal{D}} \|m_J\varphi\|_{BMO^d(\mathbb{T})}^2 \|P_{\mathcal{J}_J} b_J\|_{L^2(\mathbb{T})}^2 \\
&\leq 2\|\varphi\|_\infty^2 \sum_{J\in\mathcal{D}} \|P_{\mathcal{J}_J} b_J\|_{L^2(\mathbb{T})}^2 = 2\|\varphi\|_\infty^2 \|P_\Omega b\|_2^2 \\
&\leq 2\|\varphi\|_\infty^2 |\Omega| \|b\|_{\mathrm{BMO}^d}^2.
\end{aligned}$$

Thus

$$\frac{1}{\Omega}\|P_\Omega \Delta_{R_\varphi} b(s,t)\|_2^2 \leq 2\|\varphi\|_\infty^2 \|b\|_{\mathrm{BMO}^d}^2.$$

The proof is complete. □

The conclusion of the theorem now follows from the final lemma:

Lemma 4.2.3. *$R_{\pi_\varphi} + \pi_{R_\varphi}$ defines a bounded linear operator from* $\mathrm{BMO}(\mathbb{T}^2)$ *to* $\mathrm{BMO}^d(\mathbb{T}^2)$, *if and only if $\varphi \in \mathrm{lmo}^d(\mathbb{T}^2)$.*

Proof. Suppose that $R_{\pi_\varphi} + \pi_{R_\varphi}$ is bounded on $\mathrm{BMO}(\mathbb{T}^2) \to \mathrm{BMO}^d(\mathbb{T}^2)$. Let I be a dyadic interval and let $b_1 \in \mathrm{BMO}^d(\mathbb{T})$, $\|b_1\|_{\mathrm{BMO}(\mathbb{T})} = 1$. Define $b(s,t) = h_I |I|^{1/2} b_1(t)$, so

$$b_{I',J'} = \begin{cases} |I|^{1/2} b_{1J'} & \text{if} \quad I = I' \\ 0 & \text{otherwise,} \end{cases}$$

and $\|b\|_{\mathrm{BMO}(\mathbb{T}^2)} \approx \|b_1\|_{\mathrm{BMO}(\mathbb{T})}$. Then

$$\begin{aligned}(R_{\pi_\varphi} + \pi_{R_\varphi})b &= \sum_{J'} h_I(s)h_{J'}(t)m_I\varphi_{J'}m_{J'}b_I + \sum_{J',I'\subsetneq I} h_{I'}(s)h_{J'}(t)m_{J'}\varphi_{I'}m_{I'}b_{J'} \\ &= \sum_{J'} h_I(s)h_{J'}(t)|I|^{1/2}m_I\varphi_J m_J b_1 + \sum_{J',I'\subsetneq I} h_{I'}(s)h_{J'}(t)m_{J'}\varphi_{I'}m_{I'}b_{J'}\end{aligned}$$

Since the Haar supports of the two terms are disjoint, we know that both terms are in $\mathrm{BMO}^d(\mathbb{T}^2)$, with norm controlled by a constant $C \leq \|R_{\pi_\varphi} + \pi_{R_\varphi}\|_{\mathrm{BMO}\to\mathrm{BMO}^d}$. Let us consider the first term and fix a dyadic interval $J \in \mathbb{T}$. We obtain

$$\begin{aligned}C^2|I||J| &\geq \|P_{I\times J}\left(\sum_{J'} h_I(s)h_{J'}(t)|I|^{1/2}m_I\varphi_J m_J b_1\right)\|_2^2 \\ &= \|\sum_{J'\subseteq J} h_I(s)h_{J'}(t)|I|^{1/2}m_I\varphi_J m_J b_1\|_2^2 \\ &= |I|\|\sum_{J'\subseteq J}(m_I\varphi)_J m_J b_1 h_{J'}(t)\|_2^2 = |I|\|P_J\pi_{m_I\varphi}b_1\|^2.\end{aligned}$$

Since this estimate holds for each $b_1 \in \mathrm{BMO}(\mathbb{T})$ and each dyadic interval J, it follows that $\pi_{m_I\varphi} : \mathrm{BMO}(\mathbb{T}) \to \mathrm{BMO}^d(\mathbb{T})$ is bounded, and by Proposition 4.1.2

$$\|m_I^{(1)}\varphi\|_{\mathrm{LMO}^d(\mathbb{T})} \lesssim \|\pi_{m_I\varphi}\|_{\mathrm{BMO}(\mathbb{T})\to\mathrm{BMO}^d(\mathbb{T})} \leq C.$$

Similarly, one shows that

$$\|m_J^{(2)}\varphi\|_{\mathrm{LMO}^d(\mathbb{T})} \lesssim \|\pi_{m_J\varphi}\|_{\mathrm{BMO}(\mathbb{T})\to\mathrm{BMO}^d(\mathbb{T})} \leq C.$$

Since this estimate holds uniformly for all dyadic intervals I, J, it follows that $\varphi \in \mathrm{lmo}^d(\mathbb{T}^2)$. $\square$

Now, let us suppose that $\varphi \in \mathrm{lmo}^d(\mathbb{T}^2)$ and prove that both π_{R_φ} and R_{π_φ} are bounded. Because of the symmetry of variables, it suffices to prove this for R_{π_φ}. We recall that

$$\pi_{R_\varphi}b(s,t) = \sum_{I,J} m_J(\varphi_I)m_I(b_J)h_I(s)h_J(t).$$

Following the ideas of Theorem 3.3.9, it suffices to prove the following lemma.

Lemma 4.2.4. *Let $\varphi \in \mathrm{lmo}^d(\mathbb{T}^2)$ and $b \in \mathrm{BMO}^d(\mathbb{T}^2)$, $i, k \in \mathbb{N}$. Then*

$$\|\pi\left[\pi_R[Q_i^{(1)}\varphi]b\right]E_k^{(1)}\|_{L^2\to L^2} \lesssim \frac{k}{i}\|\varphi\|_{\mathrm{lmo}^d}\|b\|_{\mathrm{BMO}^d}.$$

Proof. We write Q for $Q^{(1)}$ and E for $E^{(1)}$. Let $\tilde{\varphi}$ be defined by

$$\tilde{\varphi}_{I,J} = \begin{cases} \varphi_{I,J} & \text{if} \quad |I| > 2^{-k} \\ (\sum_{I'\subseteq I}|\varphi_{I',J}|^2)^{1/2} & \text{if} \quad |I| = 2^{-k} \\ 0 & \text{otherwise.}\end{cases}$$

Then it follows as in the proof of Lemma 3.3.11 that

$$\|\pi_\varphi E_k^{(1)}\|_{L^2\to L^2} = \|\pi_{\tilde\varphi}\|.$$

We can assume that $\varphi = Q_i\varphi$. Following the above remark, we estimate

$$\|\widetilde{\pi_{R\varphi}b}\|_{\mathrm{BMO}^d} \leq \|\widetilde{\pi_{RE_k\varphi}}b\|_{\mathrm{BMO}^d} + \|\widetilde{\pi_{RQ_k\varphi}}b\|_{\mathrm{BMO}^d}.$$

We first consider the second term and remember that still $Q_iQ_k\varphi = Q_k\varphi$. Since $\pi_R[Q_k\varphi]b$ has no nontrivial Haar terms in the first variable for intervals I with $|I| > 2^{-k}$,

$$\widetilde{\pi_R[\varphi]}b = \sum_{J\in\mathcal{D}}\sum_{|I|=2^{-k}} h_I(s)(\sum_{I'\subseteq I}|m_{I'}\varphi_J|^2|m_Jb_{I'}|^2)^{1/2}h_J(t),$$

and this has only nontrivial Haar terms in the first variable for intervals I with $|I| = 2^{-k}$. The computation of the BMO^d norm can therefore be done by only considering rectangles of the form $S = I\times J$, $|I| = 2^{-k}$. Thus, using the one dimensional result on the boundedness of the paraproduct π on $L^2(\mathbb{T})$, the definition of $\mathrm{lmo}^d(\mathbb{T}^2)$ and Lemma 3.3.5, we obtain

$$\begin{aligned}
\|P_S(\pi_R\widetilde{[Q_k\varphi]}b)\|_2^2 &= \|\sum_{J'\subseteq J}\sum_{I'\subseteq I} h_{I'}(s)h_{J'}(t)m_{I'}b_{J'}m_{J'}\varphi_{I'}\|_2^2 \\
&= \sum_{J'\subseteq J}\|\sum_{I'\subseteq I} h_{I'}(s)m_{I'}b_{J'}m_{J'}\varphi_{I'}\|_{L^2(\mathbb{T})}^2 \\
&= \sum_{J'\subseteq J}\|\pi_{P_Im_{J'}\varphi}(\chi_I(s)b_{J'}(s))\|_{L^2(\mathbb{T})}^2 \\
&\lesssim \sum_{J'\subseteq J}\|P_Im_{J'}\varphi\|_{\mathrm{BMO}^d(\mathbb{T})}\|\chi_I(s)b_{J'}(s)\|_{L^2(\mathbb{T})}^2 \\
&\lesssim \frac{1}{i^2}\|\varphi\|_{\mathrm{lmo}^d}^2\sum_{J'\subseteq J}\|\chi_I(s)b_{J'}(s)\|_{L^2(\mathbb{T})}^2. \\
&\lesssim \frac{1}{i^2}\|\varphi\|_{\mathrm{lmo}^d}^2\|\chi_I(s)P_Jb(s,t)\|_{L^2(\mathbb{T}^2)}^2 \lesssim \frac{k^2}{i^2}\|\varphi\|_{\mathrm{lmo}^d}^2\|b\|_{\mathrm{BMO}^d(\mathbb{T}^2)}^2.
\end{aligned}$$

Let us now move to the first term $\|\widetilde{\pi_{RE_k\varphi}}b\|_{\mathrm{BMO}^d(\mathbb{T}^2)}$. Again, for we recall that still $E_k\varphi = Q_iE_k\varphi$ with this notation. Let $\Omega\subseteq\mathbb{T}^2$ be open and write $\mathcal{J}_I = \cup_{J\in\mathcal{D},I\times J\subseteq\Omega}J$ for $I\in\mathcal{D}$. We also write $\Omega_J = \cup_{I\in\mathcal{D},|I|>2^{-k},I\times J\subseteq\Omega}I$. Then using the one dimensional boundedness result of the paraproduct π on $L^2(\mathbb{T})$, the definition of $\mathrm{lmo}^d(\mathbb{T}^2)$ and Lemma

3.3.5, we obtain

$$\begin{aligned}
\|P_\Omega(\widetilde{\pi_{R_\varphi} b})\|_2^2 &= \|P_\Omega \sum_{I\in\mathcal{D},|I|>2^{-k}} \sum_{J\in\mathcal{D}} h_I(s)h_J(t)m_J\varphi_I m_I b_J\|_2^2 \\
&= \sum_{J\in\mathcal{D}} \| \sum_{I\in\mathcal{D},|I|>2^{-k},I\times J\in\Omega} h_I(s)m_J\varphi_I m_I b_J\|_2^2 \\
&= \sum_{J\in\mathcal{D}} \|P_{\Omega_J}\pi_{E_k^{(1)}m_J\varphi}\left(\sum_{I\in\mathcal{D}} h_I(s)b_{I\times J}\right)\|_2^2 \\
&= \sum_{J\in\mathcal{D}} \|P_{\Omega_J}\pi_{E_k^{(1)}m_J\varphi}(b_J)\|_2^2 \\
&= \sum_{J\in\mathcal{D}} \|P_{\Omega_J}\pi_{E_k^{(1)}m_J\varphi}(\chi_{\Omega_J}b_J)\|_2^2 \\
&= \sum_{J\in\mathcal{D}} \|P_{\Omega_J}\pi_{E_k^{(1)}m_J\varphi}(\sum_{|I|=2^{-k}} \chi_I(s)\chi_{\Omega_J}(s)b_J(s))\|_2^2 \\
&= \sum_{J\in\mathcal{D}} \|P_{\Omega_J}\pi_{E_k^{(1)}m_J\varphi}(\sum_{|I|=2^{-k},I\times J\in\Omega} \chi_I b_J)\|_2^2 \\
&\leq \sum_{J\in\mathcal{D}} \|E_k^{(1)}m_J\varphi\|^2_{\mathrm{BMO}^d(\mathbb{T})}\|\sum_{|I|=2^{-k},I\times J\in\Omega} \chi_I b_J\|_2^2 \\
&\leq \frac{1}{i^2}\|\varphi\|^2_{\mathrm{lmo}^d}\sum_{J\in\mathcal{D}}\|\sum_{I\in\mathbb{D},|I|=2^{-k},I\times J\in\Omega}\chi_I b_J\|_2^2 \\
&= \frac{1}{i^2}\|\varphi\|^2_{\mathrm{lmo}^d}\|\sum_{I\in\mathbb{D},|I|=2^{-k},I\times J\in\Omega}\chi_I(s)\sum_{J\in\mathcal{D}} b_J(s)h_J(t)\|_2^2 \\
&= \frac{1}{i^2}\|\varphi\|^2_{\mathrm{lmo}^d}\sum_{I\in\mathbb{D},|I|=2^{-k}}\|\chi_I P_{\mathcal{J}_I}b\|_2^2 \\
&\lesssim \frac{k^2}{i^2}\|\varphi\|^2_{\mathrm{lmo}^d}\|b\|^2_{\mathrm{BMO}^d}\sum_{I\in\mathbb{D},|I|=2^{-k}}|I||\mathcal{J}_I| \lesssim \frac{k^2}{i^2}\|\varphi\|^2_{\mathrm{lmo}^d}\|b\|^2_{\mathrm{BMO}^d}|\Omega|
\end{aligned}$$

This finishes the proof of the Theorem. □

We obtain in the same way the following result.

Theorem 4.2.5. *Let* $\varphi \in L^2(\mathbb{T}^2)$ *and* $\alpha \in [0,1]\times[0,1]$. *Then* $\varphi \in \mathcal{M}(\mathrm{BMO}^{d,\alpha}(\mathbb{T}^2))$ *if and only if* $\varphi \in \mathrm{lmo}^{d,\alpha}(\mathbb{T}^2)\cap \mathrm{LMO}^{d,\alpha}(\mathbb{T}^2)\cap L^\infty(\mathbb{T}^2)$.

Corollary 4.2.6. $\mathcal{M}(\mathrm{BMO}(\mathbb{T}^2)) = \mathrm{lmo}(\mathbb{T}^2)\cap\mathrm{LMO}(\mathbb{T}^2)\cap L^\infty(\mathbb{T}^2)$.

Proof. If $\varphi \in \mathcal{M}(\mathrm{BMO}(\mathbb{T}^2),\mathrm{BMO}(\mathbb{T}^2))$, then $\varphi \in \mathcal{M}(\mathrm{BMO}(\mathbb{T}^2),\mathrm{BMO}^{d,\alpha}(\mathbb{T}^2))$ for each product dyadic grid $\mathcal{R}^\alpha$, the standard dyadic grid translated by $\alpha = (\alpha_1,\alpha_2)\in[0,1]\times[0,1]$. It follows from Theorem 4.2.1 that $\varphi \in \mathrm{lmo}^{d,\alpha}(\mathbb{T}^2)\cap\mathrm{LMO}^{d,\alpha}(\mathbb{T}^2)\cap L^\infty(\mathbb{T}^2)$ with uniformly bounded norms for all $\alpha=(\alpha_1,\alpha_2)\in[0,1]\times[0,1]$, and therefore $\varphi\in\mathrm{lmo}(\mathbb{T}^2)\cap\mathrm{LMO}(\mathbb{T}^2)\cap L^\infty(\mathbb{T}^2)$.

Conversely, suppose that $\varphi \in \mathrm{lmo}(\mathbb{T}^2) \cap \mathrm{LMO}(\mathbb{T}^2) \cap L^\infty(\mathbb{T}^2)$. Thus, $\varphi \in \mathrm{lmo}^{d,\alpha}(\mathbb{T}^2) \cap \mathrm{LMO}^{d,\alpha}(\mathbb{T}^2) \cap L^\infty(\mathbb{T}^2)$ with uniformly bounded norms for all $\alpha = (\alpha_1, \alpha_2) \in [0,1] \times [0,1]$. It follows from Theorem 4.2.1 that for any $b \in BMO(\mathbb{T}^2)$, the pointwise product function $\varphi \cdot b$ is in $BMO^{d,\alpha}(\mathbb{T}^2)$ for all $\alpha = (\alpha_1, \alpha_2) \in [0,1] \times [0,1]$. Thus,

$$\varphi \cdot b \in \bigcap_{\alpha \in [0,1]\times[0,1]} BMO^{d,\alpha} = BMO(\mathbb{T}^2).$$

The proof is complete. $\square$

Part III

Hardy-type inequalities and Analytic Besov spaces on tube domains over symmetric cones

Chapter 5

Introduction to symmetric cones

This chapter introduces symmetric cones and their analysis. We give here different notions and objects useful in this part of the thesis. Our setting is an Euclidean vector space V of finite dimension n endowed with an inner product $(\cdot|\cdot)$. Materials of this chapter are essentially from the text [40].

5.1 Convex cones

We first recall the definition of a cone.

Definition 5.1.1. A subset Ω of V is said to be a cone if, for every $x \in \Omega$ and $\lambda > 0$, we have $\lambda x \in \Omega$.

A subset Ω of V is a convex cone if and only if for $x, y \in \Omega$ and $\lambda, \mu > 0$ we have $\lambda x + \mu y \in \Omega$.

Let us now move to the definition of the dual cone of a convex cone.

Definition 5.1.2. Let $\Omega \in V$ be an open convex cone. The open dual cone of Ω is defined by

$$\Omega^* = \{y \in V : (y|x) > 0, \forall x \in \overline{\Omega} \setminus \{0\}\}. \tag{5.1.1}$$

We say that Ω is self-dual whenever $\Omega^* = \Omega$.

Example 5.1.1. (1) The octant $\Omega = (0,\infty)^n$ in $V = \mathbb{R}^n$ (endowed with the canonical inner product);

(2) The Lorentz cone in $V = \mathbb{R}^n$, when $n \geq 3$

$$\Lambda_n = \left\{y \in \mathbb{R}^n \ : \ y_1^2 - y_2^2 - \ldots - y_n^2 > 0, \ \ y_1 > 0\right\}.$$

is a self-dual cone (see [40, pp. 7-10]).

5.2 Symmetric cones and Euclidean Jordan Algebras

5.2.1 Jordan algebras

We recall that a vector space V is called an algebra over $\mathbb{R}$ if a bilinear mapping $(x, y) \mapsto xy$ from $V \times V$ into V is defined. For an element $x \in V$ let $L(x)$ be the linear map of V defined by

$$L(x)y = xy.$$

An algebra V over $\mathbb{R}$ is said to be a Jordan algebra if for all elements x and y in V:

$$xy = yx, \tag{5.2.1}$$

$$x(x^2y) = x^2(xy). \tag{5.2.2}$$

Property (5.2.2) is called power associativity because if we suppose that V has an identity element $\mathbf{e}$ then (5.2.2) implies that the algebra $\mathbb{R}[x]$ generated by an element x and $\mathbf{e}$ is associative. Here,

$$\mathbb{R}[x] = \{p(x) : p \in \mathbb{R}[X]\},$$

where $\mathbb{R}[X]$ denotes the algebra over $\mathbb{R}$ of polynomials in one variable with coefficients in $\mathbb{R}$. Given $x \in V$ we have

$$\mathbb{R}[x] \sim \mathbb{R}[X]/\mathcal{I}(x),$$

where $\mathcal{I}(x)$ is the ideal defined by

$$\mathcal{I}(x) = \{p \in \mathbb{R}[X] : p(x) = 0\}.$$

Example 5.2.1. Let $M(m, \mathbb{R})$ be the algebra of $m \times m$ matrices with entries in $\mathbb{R}$ and $V = Sym(m, \mathbb{R})$ its subspace of symmetric matrices. Then V equipped with the Jordan product

$$x \circ y = \frac{1}{2}(xy + yx) \tag{5.2.3}$$

is a Jordan algebra.

Note that the powers of x cannot all be linearly independent and consequently, $\mathcal{I}(x) \neq \emptyset$. Since $\mathbb{R}[X]$ is a principal ring, $\mathcal{I}(x)$ is generated by a monic polynomial f which we call the minimal polynomial of x.

For an element $x \in V$ let $m(x)$ be the degree of the minimal polynomial of x. We have

$$m(x) = \min\{k > 0 : (\mathbf{e}, x, x^2, \cdots, x^k) \text{ are linearly dependent}\}$$

(see [40, Ch. II, pp. 28]). We have $1 \leq m(x) \leq \dim V$. We define the rank r of V as

$$r = \max\{m(x) : x \in V\},$$

and an element x is said to be regular if $m(x) = r$.

Let $L_0(x)$ be the restriction of $L(x)$ to $\mathbb{R}[x]$. For x regular, we denote by $M_0(x)$ the matrix of $L_0(x)$ with respect to the basis $\mathbf{e}, x, \cdots, x^{r-1}$. The determinant of x, $\det(x)$ and trace of x, $\operatorname{tr}(x)$ are defined by

$$det(x) = Det M_0(x)$$

and

$$tr(x) = Tr M_0(x).$$

Example 5.2.2. Let V be the space of $m \times m$ matrices with the Jordan product (5.2.3). The rank of V is m, the trace and determinant are the usual ones.

Recall that $c \in V$ is said to be an idempotent if $c^2 = c$. Two idempotents c and d are said to be orthogonal if $cd = 0$. We say that an idempotent is primitive if it is non-zero and cannot be written as the sum of two non-zero idempotents. We say that $c_1, \cdots, c_m$ is a complete system of orthogonal idempotents if

- $c_i^2 = c_i$,
- $c_i c_j = 0$ if $i \neq j$,
- $\sum_{j=1}^m c_j = \mathbf{e}$ (here $\mathbf{e}$ is the identity of V).

A complete system of idempotents is a Jordan frame if each of these idempotents is primitive. The following spectral decomposition theorem is from [40, Th. III.1.2].

Theorem 5.2.1. *(Spectral theorem). Suppose that V has rank r. Then for x in V there exists a Jordan frame $c_1, \cdots, c_r$ and real numbers $\lambda_1, \cdots, \lambda_r$ such that*

$$x = \sum_{j=1}^r \lambda_j c_j.$$

The numbers λ_j (with their multiplicities) are uniquely determined by x. Furthermore,

$$det(x) = \prod_{j=1}^r \lambda_j, \quad tr(x) = \sum_{j=1}^r \lambda_j.$$

Remark 5.2.2. The numbers λ_j in the above theorem are also called eigenvalues of x.

We end this subsection with the following definition of an Euclidean Jordan algebra.

Definition 5.2.3. A Jordan algebra V over $\mathbb{R}$ is said to be Euclidean if there exists a positive definite symmetric bilinear form on V which is associative; that is there exists on V an inner product denoted by $(u|v)$ such that $(L(x)u|v) = (u|L(x)v)$ for all x, u, v in V.

Example 5.2.3. The algebra $Sym(m, \mathbb{R})$ of $m \times m$ real symmetric matrices with the Jordan product (5.2.3) is Euclidean. In fact the bilinear form $tr(xy)$ is positive definite and associative (see [40, Ch. III]).

5.2.2 Structure of symmetric cones

Let Ω be a fixed open convex cone in V. Let $Gl(V)$ denotes the group of all linear invertible transformations of $\mathbb{R}^n$. The automorphism group $G(\Omega)$ of Ω is defined by

$$G(\Omega) = \{g \in Gl(V) : g\Omega = \Omega\}.$$

This leads to the following definition.

Definition 5.2.4. An open convex cone Ω is said to be homogeneous if the group $G(\Omega)$ acts transitively on Ω, i.e. for all $x, y \in \Omega$, there exists $g \in G(\Omega)$ such that $y = gx$. An open convex cone is said to be symmetric if it is homogeneous and self-dual.

Definition 5.2.5. A symmetric cone Ω in a Euclidean space V is said to be irreducible if there do not exist non-trivial subspaces V_1, V_2, and symmetric cones $\Omega_1 \subset V_1$, $\Omega_2 \subset V_2$, such that V is the direct sum of V_1 and V_2, and $\Omega = \Omega_1 + \Omega_2$.

We denote by G the connected component of the identity in $G(\Omega)$. G is a closed subgroup of $G(\Omega)$. If V has rank r, the determinant function satisfies the following relation

$$det(gx) = (Detg)^{\frac{r}{n}} det(x), \ x \in V, \ \ g \in G$$

(see [40, Ch. III, pp. 56]).

Let K be the subgroup of G defined by

$$K := G \cap \mathcal{O}(V)$$

where $\mathcal{O}(V)$ is the orthogonal group of V. We refer to [40, Ch. I and VI] for the following result.

Theorem 5.2.6. *Let Ω be a symmetric cone in V. Then the following are satisfied.*

(i) The identity component G of $G(\Omega)$ acts transitively on Ω.

(ii) There exists a point $\mathbf{e} \in \Omega$ such that

$$\{g \in G : g\mathbf{e} = \mathbf{e}\} = K = G(\Omega) \cap \mathcal{O}(V).$$

(iii) There exists a subgroup H of G which acts simply transitively on Ω; i.e., for all $y \in \Omega$ we can find a unique $h \in H$ such that $y = h\mathbf{e}$. Moreover, $G = HK$.

5.2.3 The Jordan algebra associated with a symmetric cone

Let Ω be a symmetric cone in a Euclidean space V. Let $\mathbf{e}$ be a point in Ω whose stabilizer is $K = G \cap \mathcal{O}(V)$. We write $\mathbf{g}$ for the Lie algebra of G. Then, following [40, Ch. III], there exists a Lie subalgebra $\mathbf{p}$ of $\mathbf{g}$ such that the mapping from $\mathbf{p}$ into V defined by

$$X \mapsto X \cdot \mathbf{e} \tag{5.2.4}$$

is a bijection. We denote by L its inverse: for $x \in V$, $L(x)$ is the unique element in $\mathbf{p}$ such that $L(x)\mathbf{e} = x$.

Theorem 5.2.7. *Let Ω be a symmetric cone in a Euclidean vector space V. Defining on V the product*

$$xy = L(x)y,$$

V is a Euclidean Jordan algebra with identity element $\mathbf{e}$ and

$$\overline{\Omega} = \{x^2 : x \in V\}.$$

A Jordan algebra is said to be simple if it does not contain any non-trivial ideal. In a simple Jordan algebra V every associative symmetric bilinear form is a scalar multiple of $tr(xy)$.

Definition 5.2.8. Let V be a simple Euclidean Jordan algebra with rank r. The rank $rk(x)$ of an element $x \in V$ is the number of non-zero eigenvalues in its spectral decomposition (with multiplicities counted).

We observe with [40, Ch. IV] that for all $x \in \Omega$, we have $rk(x) = r$. Consequently, we also call r the rank of the cone Ω.

Let us fix a Jordan frame $\{c_1, \cdots, c_r\}$ in $V = \mathbb{R}^n$. This induces a Peirce decomposition:

$$V = \bigoplus_{1 \le i \le j \le r} V_{i,j},$$

which formally lets us regard V as a space of symmetric matrices (with $V_{i,j}$ as (i,j)-entry; see [40, Ch.IV]). More precisely, the subspaces in the above decomposition are given by $V_{i,i} = \mathbb{R} \cdot c_i$ and $V_{i,j} = V(c_i, \frac{1}{2}) \cap V(c_j, \frac{1}{2}) = \{x \in V : c_i x = c_j x = \frac{x}{2}\}$ for $i < j$. For each $i < j$, the dimension of $V_{i,j}$ is the constant integer $d = 2\frac{n/r-1}{r-1}$ (see [40, pp. 71]). We denote by P_{ij} the orthogonal projection of V onto V_{ij} for $i \le j$.

We choose as H the subgroup of matrices $h \in G$ which are lower triangular with respect to the Peirce decomposition of $\mathbb{R}^n$. More precisely, H is formed by the elements $h \in G$ which satisfy the following two conditions:

(a) $h(V_{ij}) \subseteq \oplus_{(k,l) \ge (i,j)} V_{kl}$,

(b) There exist strictly positive real numbers λ_{ij}, $1 \le i \le j \le r$, such that $P_{ij} h P_{ij} = \lambda_{ij} P_{ij}$.

In assertion (a) above, $(k,l) \ge (i,j)$ denotes the lexicographic order. Then by Theorem VI.3.6 of [40], H acts simply transitively on Ω. Furthermore, one can write $H = NA = AN$, where N denotes the strictly triangular subgroup of H (i.e., matrices such that $\lambda_{ij} = 1$), and A is the diagonal subgroup. It is known from [40] that

$$A = \{P(a) : a = \sum_{1 \le i \le r} a_i c_i,\ a_i > 0\},$$

where P is the quadratic representation of V given by

$$P(x) = 2(L(x))^2 - L(x^2),\ \ x \in V.$$

It follows that

$$G = NAK = KAN.$$

We shall denote by $\Delta_1(x), \cdots, \Delta_r(x)$ the principal minors of $x \in V$, with respect to the fixed Jordan frame $\{c_1, \cdots, c_r\}$. That is, $\Delta_k(x)$ is the determinant of the projection $P_k x$ of x in the Jordan subalgebra $V^{(k)} = \oplus_{1 \le i \le j \le k} V_{i,j}$. It is well-known that

$$\Omega = \{x \in V : \Delta_k(x) > 0, k = 1, \cdots, r\}$$

(see [38, Ch. I])

Remark 5.2.9. Let us make the following observations (see [40]).

- We have the following homogeneity of the polynomials Δ_k:

$$\Delta_k(mx) = \Delta_k(x)$$

  for x in V and for any positive integer m.

- Let $x \in \Omega$, $h = nP(a) \in H$, $j \in \{1, \cdots, r\}$. Then

$$\Delta_j(hx) = \Delta_j(h\mathbf{e})\Delta_j(x) = a_1^2 \cdots a_j^2 \Delta_j(x).$$

  In particular, Δ_j is invariant under N.

-

$$\Delta_r(gy) = \Delta_r(g\mathbf{e})\Delta_r(y) = Det(g)^{\frac{r}{n}} \Delta_r(y)$$

  where $Det(g)$ is the determinant of $g \in G(\Omega)$.

- $\Delta_r(ky) = \Delta_r(y)$ for all $k \in K$.

5.3 Integrals over Ω

We suppose in this section that Ω is an irreducible cone of rank r in $\mathbb{R}^n$. Let us introduce some standard notations on multi-indices:

1. If $t = (t_1, \cdots, t_r)$ we denote by t^* the vector $t^* = (t_r, \cdots, t_1)$,
2. For $t = (t_1, \cdots, t_r)$ and $a \in \mathbb{R}$, by convention $a + t$ denotes the multi-index defined by $(t_1 + a, \cdots, t_r + a)$,
3. For $t \in \mathbb{R}^r$ and $k \in \mathbb{R}^r$, we write that $t < k$ if $t_j < k_j$ for $j = 1, \cdots, r$.

We also introduce the particular multi-index

$$g_0 := \left((j-1)\frac{d}{2}\right)_{1 \le j \le r} \text{ with } (r-1)\frac{d}{2} = \frac{n}{r} - 1.$$

The generalized power function on Ω is defined as

$$\Delta_s(x) = \Delta_1^{s_1 - s_2}(x)\Delta_2^{s_2 - s_3}(x) \cdots \Delta_r^{s_r}(x), \; x \in \Omega; \; s \in \mathbb{C}^r.$$

Remark 5.3.1. Let $s = (s_1, \cdots, s_r) \in \mathbb{C}^r$. Then,

- $\Delta_s(a_1c_1 + \cdots + a_rc_r) = a_1^{s_1} \cdots a_r^{s_r}$ for strictly positive $a_1, \cdots, a_r$, whence the name of generalized power;

- $|\Delta_s| = \Delta_{\Re s}$

We now recall the definition of generalized gamma function on Ω:

$$\Gamma_\Omega(s) = \int_\Omega e^{-(\mathbf{e}|\xi)} \Delta_s(\xi) \Delta^{-n/r}(\xi) d\xi \quad s = (s_1, \cdots, s_r) \in \mathbb{C}^r.$$

This integral converges if and only if $\Re s_j > (j-1)\frac{n/r-1}{r-1} = (j-1)\frac{d}{2}$ for all $j = 1, \cdots, r$, being in this case equal to:

$$\Gamma_\Omega(s) = (2\pi)^{\frac{n-r}{2}} \prod_{j=1}^{r} \Gamma(s_j - (j-1)\frac{d}{2}),$$

where Γ is the usual gamma function on the positive half real line $\mathbb{R}_+$ (see Chapter VII of [40]). For $s \in \mathbb{R}$, we write $\Gamma_\Omega((s, \cdots, s)) = \Gamma_\Omega(s)$. We have the following result on the Laplace transform of the generalized power function (see Proposition VII.1.2 and Proposition VII.1.6 in [40]).

Lemma 5.3.2. *Let $s = (s_1, \cdots, s_j) \in \mathbb{C}^r$ with $\Re s_j > (j-1)\frac{d}{2}$, $j = 1, \cdots, r$. Then for all $y \in \Omega$ we have*

$$\int_\Omega e^{-(y/\xi)} \Delta_s(\xi) \Delta^{-n/r}(\xi) d\xi = \Gamma_\Omega(s) \Delta_s(y^{-1}).$$

Here, $y = he$ if and only if $y^{-1} = h^{*-1}e$ with $h \in H$.

Let us remark that the power function $\Delta_s(y^{-1})$ in the above lemma can also be expressed in terms of the rotated Jordan frame $\{c_r, \cdots, c_1\}$. If we denote by Δ_j^*, $j = 1, \cdots, r$, the principal minors with respect to this new frame then

$$\Delta_s(y^{-1}) = [\Delta_{s^*}^*(y)]^{-1} \quad \forall s = (s_1, \cdots, s_r) \in \mathbb{C}^r.$$

The general power function can be extended to $T_\Omega = V + i\Omega$ via the Fourier-Laplace transform. We refer to [51] for the following.

Lemma 5.3.3. *The following assertions holds.*

(1) Let $s_1, \cdots, s_r \geq 0$ and $s = (s_1, \cdots, s_r)$. Then

$$\Delta_s(y+u) \geq \Delta_s(y), \ \forall y, u \in \Omega.$$

(2) For $s = (s_1, \cdots, s_r) > g_0$, the integral

$$\frac{1}{\Gamma_\Omega(s)} \int_\Omega e^{i(z|\xi)} \Delta_s^*(\xi) \frac{d\xi}{\Delta^{\frac{n}{r}}(\xi)}$$

is absolutely convergent and defines a holomorphic function on $V + i\Omega$. It is denoted by $\left[\Delta_s(\frac{z}{i})\right]^{-1}$.

By Corollary 2.18 and Corollary 2.19 of [38] (see also [55]) we have the following lemma.

Proposition 5.3.4. *Let $t \in \mathbb{C}^r$ and $y \in \Omega$.*

(1) The integral

$$J_t(y) = \int_{\mathbb{R}^n} \left| \Delta_{-t}(\frac{x + iy}{i}) \right| dx$$

converges if and only if $\Re t > g_0^* + \frac{n}{r}$. In this case, $J_t(y) = C_t |\Delta_{-t}(y)| \Delta^{\frac{n}{r}}(y)$, where

$$C_t = \frac{(2\pi)^{2n} 2^{n - \Re(\sum_{j=1}^r t_j)} \Gamma_\Omega(\Re t^* - \frac{n}{r})}{|\Gamma(\frac{t^*}{2})|^2}.$$

(2) For multi-indices s and t and for any $u \in \Omega$, the functions $y \mapsto \Delta_t(y + u)\Delta_s(y)$ belongs to $L^1(\Omega, \frac{dy}{\Delta^{n/r}(y)})$ if and only if $\Re s > g_0$ and $\Re(s + t) < -g_0^*$. In this case, we have

$$\int_\Omega \Delta_t(y + u) \Delta_s(y) \frac{dy}{\Delta^{n/r}(y)} = \frac{\Gamma_\Omega(s)\Gamma_\Omega(-(s + t)^*)}{\Gamma(-t^*)} \Delta_{s+t}(u).$$

Chapter 6

Boundedness of Bergman-type operators

In this chapter, we study here boundedness properties of Rudin-Forelli-type operators associated to tubular domains over symmetric cones. As an application, we give a characterization of the topological dual space of the weighted Bergman space $A_\nu^{p,q}$. We essentially make use of the methods in [10, 14].

6.1 Introduction

Let V be a real vector space of dimension n, endowed with the structure of a simple Euclidean Jordan algebra. We consider an irreducible symmetric cone Ω inside $V = \mathbb{R}^n$ and denote by $T_\Omega = V + i\Omega$ the corresponding tube domain in the complexification of V. Here, V is endowed with an inner product $(.|.)$ for which the cone Ω is self-dual. We recall that as an example of a symmetric cone in $\mathbb{R}^n$, we have the forward light cone given for $n \geq 3$ by

$$\Gamma_n = \{y \in \mathbb{R}^n : y_1^2 - y_2^2 - \cdots - y_n^2 > 0, \ \ y_1 > 0\}.$$

Again, we write r for the rank of Ω and $\Delta(x)$ for the associated determinant function. Light cones have rank 2 and determinant function given by the Lorentz form

$$\Delta(y) = y_1^2 - y_2^2 - \cdots - y_n^2, \ \text{ for } \ y = (y_1, y_2, \cdots, y_n).$$

We recall that given $1 \leq p, q < \infty$ and $\nu \in \mathbb{R}$, the mixed norm Lebesgue space $L_\nu^{p,q}(T_\Omega)$

is defined by the integrability condition

$$||f||_{L^{p,q}_\nu} := \left[\int_\Omega \left(\int_{\mathbb{R}^n} |f(x+iy)|^p dx\right)^{\frac{q}{p}} \Delta^{\nu-\frac{n}{r}}(y)dy\right]^{\frac{1}{q}} < \infty. \tag{6.1.1}$$

The mixed norm weighted Bergman space $A^{p,q}_\nu(T_\Omega)$ is then the closed subspace of $L^{p,q}_\nu(T_\Omega)$ consisting of holomorphic functions on the tube T_Ω. These spaces are nontrivial only when $\nu > \frac{n}{r} - 1$ (see [12]). When $p = q$ we shall simply write $A^{p,p}_\nu = A^p_\nu$. The usual Bergman space A^p then corresponds to the case $\nu = \frac{n}{r}$.

The weighted Bergman projection P_ν is the orthogonal projection from the Hilbert space $L^2_\nu(T_\Omega)$ onto its closed subspace $A^2_\nu(T_\Omega)$ and it is given by the integral formula

$$P_\nu f(z) = \int_{T_\Omega} B_\nu(z,w) f(w) \Delta^{\nu-\frac{n}{r}}(\Im w) dV(w), \tag{6.1.2}$$

where

$$B_\nu(z,w) = d_\nu \Delta^{-\nu-\frac{n}{r}}\left(\frac{z-\overline{w}}{i}\right) \tag{6.1.3}$$

is the weighted Bergman kernel, $d_\nu = \frac{2^{r\nu}}{(2\pi)^n}\frac{\Gamma_\Omega(\nu+\frac{n}{r})}{\Gamma_\Omega(\nu)}$ and dV is the Lebesgue measure on $\mathbb{C}^n$(see [12]). Let us recall that the Bergman kernel is a reproducing kernel on $A^2_\nu(T_\Omega)$, that is for any $f \in A^2_\nu(T_\Omega)$, $z \in T_\Omega$,

$$f(z) = c\int_{T_\Omega} B_\nu(z,w) f(w) \Delta^{\nu-\frac{n}{r}}(\Im w) dV(w). \tag{6.1.4}$$

The $L^{p,q}_\nu$-boundedness of the Bergman projection P_ν is still an open problem and has attracted a lot of attention in recent years (see [14], [10], [9], [13]). To date, it is only known that this projection extends to a bounded operator on $L^{p,q}_\nu$ for general symmetric cones for the range $1 \le p < \infty$ and $q'_{\nu,p} < q < q_{\nu,p}$, with $q_{\nu,p} = \min\{p, p'\} q_\nu$, $q_\nu = 1 + \frac{\nu}{\frac{n}{r}-1}$ and $\frac{1}{p} + \frac{1}{p'} = 1$ (see for example [13]) with slight improvements over this range in the case of light cones (see [52]).

The importance of the boundedness of the Bergman projection can be expressed in terms of its consequences, among which the following is a well-known one: If P_ν extends to a bounded operator on $L^{p,q}_\nu$, then the topological dual space $(A^{p,q}_\nu)^*$ of the Bergman space $A^{p,q}_\nu$ identifies with $A^{p',q'}_\nu$ under the integral pairing

$$\langle f, g\rangle_\nu = \int_{T_\Omega} f(z)\overline{g(z)} \Delta^{\nu-n/r}(\Im z) dV(z),$$

$f \in A^{p,q}_\nu$ and $g \in A^{p',q'}_\nu$ (see [12]). So, since the range of boundedness of P_ν on $L^{p,q}_\nu$ is far from being completely known, a natural question is: is there any way of characterizing the dual space of $A_\nu^{p,q}$ for values of the parameters $p,\ q,\ \nu$ for which P_ν is not necessarily bounded? To answer this type of question, it seems natural to consider the problem of $L_\nu^{p,q}$-boundedness of a family of operators generalizing the Bergman projection. This family is given by the integral operators $T = T_{\alpha,\beta,\gamma}$ and $T^+ = T^+_{\alpha,\beta,\gamma}$ defined on $C_c^\infty(T_\Omega)$ by the formulas

$$Tf(z) = \Delta^\alpha(\Im z)\int_{T_\Omega} B_\gamma(z,w)f(w)\Delta^\beta(\Im w)dV(w) \tag{6.1.5}$$

and

$$T^+f(z) = \Delta^\alpha(\Im z)\int_{T_\Omega} |B_\gamma(z,w)|f(w)\Delta^\beta(\Im w)dV(w). \tag{6.1.6}$$

Remark 6.1.1. The boundedness of T^+ on $L_\nu^{p,q}(T_\Omega)$ implies the boundedness of T, although the boundedness of T is typically expected on a larger range than T^+.

The boundedness of this family of operators on $L_\nu^{p,q}(T_\Omega)$ has been considered in [14] for the case $P_\mu = T_{0,\mu-\frac{n}{r},\mu}$ and in [10] for $T_{0,\mu-\frac{n}{r},\mu+m}$. Both works deal with the case of the light cone. Here, we consider the problem of the boundedness of the operator T^+ for general symmetric cones and obtain optimal results for this operator. For this, we systematically make use of the methods of [14] and [10] which seem to be appropriate here. Since we are considering general symmetric cones, the general power function defined in the text is also useful in this case. We mention that the case $p = q$ for general symmetric cones was implicit in [17].

As an application, we characterize the dual space of Bergman spaces in some cases where the Bergman projection is not necessarily bounded, answering partially the question mentioned above.

6.2 Integral operators on the cone

The aim of this section is to give L_ν^q-continuity properties of a family of operators on the cone Ω which are closely related to the operators $T_{\alpha,\beta,\gamma}$. Considering $V = \mathbb{R}^n$ as a Jordan algebra, we denote its identity element by $\mathbf{e}$ (this correspond to the point (1,0,...,0) in the forward light cone). We recall that a generalized power in the symmetric cone Ω of rank r is defined by

$$\Delta_s(x) = \Delta_1^{s_1-s_2}(x)\Delta_2^{s_2-s_3}(x)\cdots\Delta_r^{s_r}(x), s = (s_1, s_2, \cdots, s_r) \in \mathbb{C}^r,$$

where $x \in \Omega$ and $\Delta_k(x)$ are the principal minors of x as defined in the previous chapter.

We now recall Schur's lemma.

Lemma 6.2.1 (Schur's Lemma). *Let μ be a positive measure on a measure space X, let $H(x,y)$ be a positive measurable function on $X \times X$, and let $q > 1$, with $\frac{1}{q} + \frac{1}{q'} = 1$. If there exists a positive measurable function $h(x)$ on X and a positive constant C such that*

$$\int_X H(x,y)h^q(x)d\mu(x) \leq Ch^q(y)$$

and

$$\int_X H(x,y)h^{q'}(y)d\mu(y) \leq Ch^{q'}(x)$$

for all x and y in X, then the integral operator

$$Hf(x) = \int_X H(x,y)f(y)d\mu(y)$$

is bounded on $L^q(X,\mu)$ with $||H|| \leq C$.

Proof. See [114] Theorem 3.2.2. □

For real parameters α, β, γ, we now consider the integral operators $S = S_{\alpha,\beta,\gamma}$ which are defined on the cone Ω by

$$Sg(y) = \int_\Omega \Delta^\alpha(y)\Delta^{-\gamma}(y+v)g(v)\Delta^\beta(v)dv. \tag{6.2.1}$$

The following lemmas give continuity properties of the operators $S_{\alpha,\beta,\gamma}$ on $L^q_\nu(\Omega) = L^q(\Omega, \Delta^{\nu-\frac{n}{r}}(y)dy)$, $\nu \in \mathbb{R}$.

Lemma 6.2.2. *Let $\nu \in \mathbb{R}$, $1 < q < \infty$, $\gamma = \alpha + \beta + \frac{n}{r}$ with*

$$\max\{-q\alpha + \frac{n}{r} - 1, q(-\alpha + \frac{n}{r} - 1) - \frac{n}{r} + 1\} < \nu < \min\{q(\beta+1) + \frac{n}{r} - 1, q(\beta + \frac{n}{r}) - \frac{n}{r} + 1\}.$$

Then the operator $S = S_{\alpha,\beta,\gamma}$ is bounded on $L^q(\Omega, \Delta^{\nu-\frac{n}{r}}(y)dy)$.

Proof. We can write the integral S as

$$Sg(y) = \int_\Omega H(y,v)g(v)\Delta^{\nu-\frac{n}{r}}(v)dv,$$

where $H(y,v) = \Delta^\alpha(y)\Delta^{-\gamma}(y+v)\Delta^{\beta-\nu+\frac{n}{r}}(v)$ is a positive kernel with respect to the measure $\Delta^{\nu-\frac{n}{r}}(v)dv$. By Schur's lemma, it is sufficient to find a positive function h on Ω such that

$$\int_\Omega H(x,y)h^{q'}(y)d\mu(y) \leq Ch^{q'}(x)$$

and

$$\int_\Omega H(x,y)h^q(x)d\mu(x) \leq Ch^q(y)$$

for $q > 1$ and $d\mu(y) = \Delta^{\nu-\frac{n}{r}}(y)dy$. We take $h(y) = \Delta_s(y)$, where $s = (s_1, \cdots, s_r)$ and $s_j, j = 1, \cdots, r$, are real numbers to be determined.

Straightforward computations with the use of the given choice of h and Proposition 5.3.4 yield

$$\frac{-\alpha-\nu+(j-1)\frac{d}{2}}{q'} < s_j < \frac{-\alpha-\nu+\gamma-\frac{n}{r}+1+(j-1)\frac{d}{2}}{q'}$$

and

$$\frac{-\beta-\frac{n}{r}+(j-1)\frac{d}{2}}{q} < s_j < \frac{-\beta+\gamma+(j-1)\frac{d}{2}-2\frac{n}{r}+1}{q}.$$

Thus, each s_j must belong to an intersection of two intervals. This intersection is not empty by the hypotheses, since the condition

$$\max\{-q\alpha+\frac{n}{r}-1, q(-\alpha+\frac{n}{r}-1)-\frac{n}{r}+1\} < \nu < \min\{q(\beta+1)+\frac{n}{r}-1, q(\beta+\frac{n}{r})-\frac{n}{r}+1\}$$

is equivalent to

$$\max_{1\leq j\leq r}\{q(-\alpha+\frac{n}{r}-1-(j-1)\frac{d}{2})+(j-1)d-\frac{n}{r}+1\} < \nu < \min_{1\leq j\leq r}\{q(\beta+\frac{n}{r}-(j-1)\frac{d}{2})-\frac{n}{r}+1+(j-1)d\}.$$

It follows that S is bounded on $L^q_\nu(\Omega)$ for every $q > 1$ and the proof is complete. $\square$

Lemma 6.2.3. *Suppose $1 < q < \infty$, $\nu \in \mathbb{R}$ and $S = S_{\alpha,\beta,\gamma}$ is bounded on $L^q(\Omega, \Delta^{\nu-\frac{n}{r}}(y)dy)$. Then*

$$\max\{-q\alpha+\tfrac{n}{r}-1, q(\beta-\gamma+2\tfrac{n}{r}-1)-\tfrac{n}{r}+1\} < \nu < \min\{q(\gamma-\alpha)-\tfrac{n}{r}+1, q(\beta+1)+\tfrac{n}{r}-1\}.$$

Proof. Let us take the characteristic function of the Euclidean ball $b_1(\underline{\mathbf{e}})$ of radius 1 centered at $\underline{\mathbf{e}}$ as a test function and denote this by g. By continuity, $\Delta(v)$ is almost constant on the support of g. Let us estimate $\Delta(v+y)$ on the support of $g(v)$ for $y \in \Omega$ fixed. For that, we remark that for all y and t in the cone Ω and $\lambda > \frac{n}{r}-1$ we can write

$$\Delta^{-\lambda}(y+v) = c\int_\Omega e^{-(y+v|\xi)}\Delta^{\lambda-\frac{n}{r}}(\xi)d\xi \tag{6.2.2}$$

(see Chapter VII of [40]). By Theorem 2.45 of [12] there exists a constant $C = C(\Omega) \geq 1$ such that for all $\xi \in \Omega$,

$$\frac{1}{C} \leq \frac{(v|\xi)}{(\underline{\mathbf{e}}|\xi)} \leq C \text{ whenever } v \in b_1(\underline{\mathbf{e}}). \tag{6.2.3}$$

Remarking that for $C > 1$, $\frac{1}{C}(y+v|\xi) \leq \frac{1}{C}(v|\xi)+(y|\xi) \leq (y+v|\xi) \leq C(v|\xi)+(y|\xi) \leq C(y+v|\xi)$, we obtain using the estimates (6.2.3), formula (6.2.2) and the fact that the determinant function is homogeneous of degree r (see [40]) that for v in the support of g and $y \in \Omega$, the following hold:

$$\begin{aligned}(\frac{1}{C})^r \Delta(\underline{\mathbf{e}}+y) &= \Delta(\frac{1}{C}(\underline{\mathbf{e}}+y)) \leq \Delta(\frac{1}{C}\underline{\mathbf{e}}+y) \\ &\leq \Delta(v+y) \leq \Delta(C\underline{\mathbf{e}}+y) \leq \Delta(C(\underline{\mathbf{e}}+y)) = C^r\Delta(\underline{\mathbf{e}}+y).\end{aligned}$$

We conclude that there exists a constant $C = C(\Omega) \geq 1$ such that, for all $y \in \Omega$,

$$\frac{1}{C}\Delta(\underline{\mathbf{e}}+y) \leq \Delta(v+y) \leq C\Delta(\underline{\mathbf{e}}+y), \quad \text{for all} \quad v \in b_1(\underline{\mathbf{e}}).$$

It follows that $Sg(y) = S\chi_{b_1(\underline{\mathbf{e}})}(y) \approx C\Delta^{\alpha}(y)\Delta^{-\gamma}(y+\underline{\mathbf{e}})$. So if S is bounded on $L^q(\Omega, \Delta^{\nu-\frac{n}{r}}(y)dy)$, then the function $\Delta^{\alpha}(y)\Delta^{-\gamma}(y+\underline{e})$ is in $L^q(\Omega, \Delta^{\nu-\frac{n}{r}}(y)dy)$, which means that

$$\int_{\Omega} \Delta^{q\alpha+\nu-\frac{n}{r}}(y)\Delta^{-q\gamma}(y+\underline{\mathbf{e}})dy < \infty.$$

By Proposition 5.3.4 we necessarily have $q\alpha+\nu-\frac{n}{r} > -1$ and $-q\gamma+q\alpha+\nu-\frac{n}{r} < -2\frac{n}{r}+1$, which is equivalent to $\nu > -q\alpha + \frac{n}{r} - 1$ and $\nu < q(\gamma-\alpha) - \frac{n}{r} + 1$ with $1 \leq q < \infty$. This gives half of the conditions.

By duality, the boundedness of S on $L^q(\Omega, \Delta^{\nu-\frac{n}{r}}(y)dy)$ implies the boundedness of its adjoint S^* on $L^{q'}(\Omega, \Delta^{\nu-\frac{n}{r}}(y)dy)$, where $\frac{1}{q}+\frac{1}{q'} = 1$. It is easy to see that

$$S^*g(y) = \int_{\Omega} \Delta^{\beta-\nu+\frac{n}{r}}(y)\Delta^{-\gamma}(y+v)g(v)\Delta^{\alpha+\nu-\frac{n}{r}}(v)dv$$

Using the same reasoning as before, we obtain that the function $\Delta^{\beta-\nu+\frac{n}{r}}(y)\Delta^{-\gamma}(y+\underline{e})$ must belong to $L^{q'}(\Omega, \Delta^{\nu-\frac{n}{r}}(y)dy)$. Again by Proposition 5.3.4, we must have $(\beta-\nu+\frac{n}{r})q'+\nu-\frac{n}{r} > -1$ and $-q'\gamma+(\beta-\nu+\frac{n}{r})q'+\nu-\frac{n}{r} < -2\frac{n}{r}+1$, which is equivalent to $\nu < q(\beta+1)+\frac{n}{r}-1$ and $\nu > q(\beta-\gamma+2\frac{n}{r}-1)-\frac{n}{r}+1$. This completes the proof of the lemma. □

Lemma 6.2.4. *For $\nu \in \mathbb{R}$, the operator $S = S_{\alpha,\beta,\gamma}$ is bounded on $L^1(\Omega, \Delta^{\nu-\frac{n}{r}}(y)dy)$ if and only if $\gamma = \alpha+\beta+\frac{n}{r}$ and $-\alpha+\frac{n}{r}-1 < \nu < \beta+1$.*

Proof. We first show the sufficient condition. For any function g in $L^1_{\nu}(\Omega)$, using Fubini's theorem, we have that

$$\begin{aligned}\int_{\Omega} |Sg(y)|\Delta^{\nu-\frac{n}{r}}(y)dy &\leq \int_{\Omega}\left(\int_{\Omega}\Delta^{\alpha}(y)\Delta^{-\gamma}(y+v)|g(v)|\Delta^{\beta}(v)dv\right)\Delta^{\nu-\frac{n}{r}}(y)dy \\ &= \int_{\Omega}|g(v)|\left(\int_{\Omega}\Delta^{-\gamma}(y+v)\Delta^{\alpha+\nu-\frac{n}{r}}(y)dy\right)\Delta^{\beta}(v)dv \\ &= C\int_{\Omega}|g(v)|\Delta^{\nu-\frac{n}{r}}(v)dv,\end{aligned}$$

where the last equality follows from Proposition 5.3.4, since $\gamma = \alpha+\beta+\frac{n}{r}$ and $-\alpha+\frac{n}{r}-1 < \nu < \beta+1$.

To prove the necessary condition, we proceed as in the proof of Lemma 6.2.3. That means, the operator

$$S^*g(y) = \int_\Omega \Delta^{\beta-\nu+\frac{n}{r}}(y)\Delta^{-\gamma}(y+v)g(v)\Delta^{\alpha+\nu-\frac{n}{r}}(v)dv$$

must be bounded on $L^\infty(\Omega)$. As test function, we take $g = 1$. Then

$$|S^*g(y)| = \int_\Omega \Delta^{\beta-\nu+\frac{n}{r}}(y)\Delta^{-\gamma}(y+v)\Delta^{\alpha+\nu-\frac{n}{r}}(v)dv.$$

It follows from Lemma 8.2.1 that we must necessarily have $\alpha+\nu-\frac{n}{r} > -1$ and $-\gamma+\alpha+\nu-\frac{n}{r} < -2\frac{n}{r}+1$. In this case, $|S^*g(y)| = C\Delta^{\beta-\gamma+\alpha+\frac{n}{r}}(y)$ for all $y \in \Omega$. Thus, S^*g belongs to $L^\infty(\Omega)$ if and only if $\beta-\gamma+\alpha+\frac{n}{r} = 0$. This completes the proof of the lemma. □

6.3 Positive integral operators on the tube $T_\Omega = V + i\Omega$

In this section, we give some boundedness conditions for the family of integral operators $T_{\alpha,\beta,\gamma}$ defined on the tube T_Ω . We begin by recalling some results.

Lemma 6.3.1. *([12], Lemma 4.11) For $\nu \in \mathbb{R}$, there are constants $C_\nu > 0$ and $\delta > 0$ such that for all $z = x+iy \in T_\Omega$, $v \in \Omega$ with $|x| \le \frac{1}{2}$, $|v|, |y| < \delta$,*

$$\int_{|u|\le 1} |B_\nu(z, u+iv)|du \ge C_\nu \Delta^{-\nu}(y+v).$$

Lemma 6.3.2. *Let α be real. Then the function $f(z) = \Delta^{-\alpha}(\frac{z+it}{i})$ with $t \in \Omega$, belongs to $A^{p,q}_\nu(T_\Omega)$ if and only if $\nu > \frac{n}{r}-1$ and $\alpha > max\left(\frac{2\frac{n}{r}-1}{p}, \frac{n}{rp} + \frac{\nu+\frac{n}{r}-1}{q}\right)$. In this case,*

$$||f||^q_{A^{p,q}_\nu} = C_{\alpha,p,q}\Delta^{-q\alpha+\frac{nq}{rp}+\nu}(t).$$

Proof. See [12], Lemma 3.20. □

Theorem 6.3.3. *Suppose $\nu \in \mathbb{R}$ and $1 \le p, q < \infty$. Then the following conditions are equivalent.*

(a) The operator T^+ defined by (6.1.6) is bounded on $L^{p,q}_\nu(T_\Omega)$.

(b) The parameters satisfy $\gamma = \alpha+\beta+\frac{n}{r}$, $\alpha+\beta > -1$ and

$$\max\{-q\alpha+\frac{n}{r}-1, q(-\alpha+\frac{n}{r}-1)-\frac{n}{r}+1\} < \nu < \min\{q(\beta+1)+\frac{n}{r}-1, q(\beta+\frac{n}{r})-\frac{n}{r}+1\}.$$

Proof. The ideas of the proof are the same as in [10] and [12]. Let us first prove the sufficient condition. For $f : T_\Omega \to \mathbb{C}$, we write $f_y(x) = f(x+iy)$. Then

$$\begin{aligned} T^+f(x+iy) = (T^+f)_y(x) &= d_\gamma \Delta^\alpha(y) \left(\int_\Omega \int_{\mathbb{R}^n} |\Delta_{y+v}^{-(\gamma+\frac{n}{r})}(x-u)| f_v(u) du \right) \Delta^\beta(v) dv \\ &= d_\gamma \Delta^\alpha(y) \int_\Omega (|\Delta_{y+v}^{-(\gamma+\frac{n}{r})}| * f_v)(x) \Delta^\beta(v) dv. \end{aligned}$$

Without loss of generality, we may assume that f is non-negative. By the Minkowski inequality, the Young inequality and part (1) of Proposition 5.3.4, we obtain

$$\begin{aligned} ||(T^+f)_y||_{L^p(\mathbb{R}^n)} &= \left(\int_{\mathbb{R}^n} |(T^+f)_y(x)|^p dx \right)^{\frac{1}{p}} \\ &= d_\gamma \Delta^\alpha(y) \left(\int_{\mathbb{R}^n} \left(\int_\Omega (|\Delta_{y+v}^{-(\gamma+\frac{n}{r})}| * f_v)(x) \Delta^\beta(v) dv \right)^p dx \right)^{\frac{1}{p}} \\ &\leq d_\gamma \Delta^\alpha(y) \int_\Omega || |\Delta_{y+v}^{-(\gamma+\frac{n}{r})}| * f_v||_p \Delta^\beta(v) dv \\ &\leq d_\gamma \Delta^\alpha(y) \int_\Omega ||\Delta_{y+v}^{-(\gamma+\frac{n}{r})}||_1 ||f_v||_p \Delta^\beta(v) dv \\ &= C_\alpha \int_\Omega \Delta^\alpha(y) \Delta^{-\gamma}(y+v) ||f_v||_p \Delta^\beta(v) dv \\ &= C_\alpha S(||f_v||_p)(y), \end{aligned}$$

where $||\Delta_{y+v}^{-(\gamma+\frac{n}{r})}||_1$ is given by part (1) of Proposition 5.3.4. The sufficient condition then follows from Lemma 6.2.2 and Lemma 6.2.4.

We now prove the necessary condition. We first show that if the operator T^+ is bounded on $L_\nu^{p,q}(T_\Omega)$, then the equality $\gamma = \alpha + \beta + \frac{n}{r}$ necessarily holds. We recall that the determinant function is homogeneous of degree r (see [40]). For $f \in L_\nu^{p,q}(T_\Omega)$, we define f_R, $R > 0$, by $f_R(z) = f(Rz)$ for any $z \in T_\Omega$. The function f_R belongs to $L_\nu^{p,q}(T_\Omega)$. Using the homogeneity of the determinant function, we obtain

$$||f_R||^q_{L_\nu^{p,q}} = R^{-r(\nu-\frac{n}{r})-n\frac{q}{p}-n} ||f||^q_{L_\nu^{p,q}}$$

and

$$||T^+(f_R)||^q_{L_\nu^{p,q}} = R^{r(\gamma+\frac{n}{r})q-r\alpha q-r(\nu-\frac{n}{r})-n\frac{q}{p}-n-q(r\beta+2n)} ||Tf||^q_{L_\nu^{p,q}}.$$

It follows from the hypotheses that there exists a positive constant C such that $||T^+(f_R)||_{L_\nu^{p,q}} \leq C||f_R||_{L_\nu^{p,q}}$. This is equivalent to $R^{r(\gamma-\alpha-\beta)-n}||T^+f||_{L_\nu^{p,q}} \leq C||f||_{L_\nu^{p,q}}$ for all $R > 0$, which necessarily implies that $\gamma = \alpha + \beta + \frac{n}{r}$. The condition $\alpha + \beta > -1$ is naturally necessary since if it does not hold, the range of ν is be empty. To obtain the other necessary conditions, we test T^+ on the functions $f(x+iy) = \chi_{|x|<1}(x) g(y)$, with g a positive function compactly supported in the intersection of the cone with the Euclidean ball of radius δ centered at 0. Using Lemma 6.3.1, it follows that for x,y with $|x| < \frac{1}{4}$, $|y| < \delta$, the following inequality holds:

$$T^+f(x+iy) \geq C\Delta^\alpha(y)\int_\Omega \Delta^{-\gamma}(y+v)g(v)\Delta^\beta(v)dv.$$

Then, by assumption, there exists a constant C independent of g such that

$$\int_{y\in\Omega,|y|<\delta}\left(\Delta^\alpha(y)\int_\Omega \Delta^{-\gamma}(y+v)g(v)\Delta^\beta(v)dv\right)^q \Delta^{\nu-\frac{n}{r}}(y)dy \leq C\int_\Omega g^q(v)\Delta^{\nu-\frac{n}{r}}(v)dv.$$

By homogeneity of the kernel, we can replace the constant δ by an arbitrary positive constant K. It follows that for every positive function g on Ω, we have the inequality

$$\int_{y\in\Omega,|y|<K}(\Delta^\alpha(y)\int_\Omega \Delta^{-\gamma}(y+v)g(v)\Delta^\beta(v)dv)^q\Delta^{\nu-\frac{n}{r}}(y)dy \leq C\int_{v\in\Omega,|v|<K} g^q(v)\Delta^{\nu-\frac{n}{r}}(v)dv$$

Then by density of compactly supported functions, we have the same inequality without any bound on the integrals. The other necessary condition of the theorem is then a consequence of the necessary conditions in Lemma 6.2.3 and Lemma 6.2.4 and the relation obtained previously, $\gamma = \alpha+\beta+\frac{n}{r}$. This completes the proof of the theorem. □

Corollary 6.3.4. *Let $1 \leq p, q < \infty$ and $\nu \in \mathbb{R}$. If $\gamma = \alpha+\beta+\frac{n}{r}$, $\alpha+\beta > -1$ and*

$$\max\{-q\alpha+\frac{n}{r}-1, q(-\alpha+\frac{n}{r}-1)-\frac{n}{r}+1\} < \nu < \min\{q(\beta+1)+\frac{n}{r}-1, q(\beta+\frac{n}{r})-\frac{n}{r}+1\},$$

then the operator $T_{\alpha,\beta,\gamma}$ is bounded on $L^{p,q}_\nu(T_\Omega)$.

We define the Berezin transform on T_Ω as the operator defined on $L^1(T_\Omega)$ by the pairing

$$\langle f\tilde{B}_z, \tilde{B}_z\rangle, \quad z \in T_\Omega,$$

where $\tilde{B}_z = \Delta^{\frac{n}{r}}(\Im z)B_{\frac{n}{r}}(.,z)$ is the normalized reproducing kernel of $A^2(T_\Omega)$ (see for example [114] for more on the Berezin transform).

Corollary 6.3.5. *Let $1 \leq p, q < \infty$. Then the Berezin transform defined on T_Ω by*

$$B(f)(z) = \Delta^{2\frac{n}{r}}(\Im z)\int_{T_\Omega}|B_{3\frac{n}{r}}(z,w)|f(w)dV(w), \quad z \in T_\Omega$$

is bounded on $L^{p,q}(T_\Omega, dV(z))$, if and only if $q > 2-\frac{r}{n}$.

Remark 6.3.6. Let us remark that the above corollary was proved in [50] in the setting of light cones and for $p = q$.

Corollary 6.3.7. *Let $1 \leq p, q < \infty$. If ν and m are real numbers such that $\nu + m > \frac{n}{r} - 1$, then the positive operator Q^+ defined by*

$$Q^+ f(z) = \int_{T_\Omega} |B_{\nu+m}(z,w)| f(w) \Delta^{\nu - \frac{n}{r}}(\Im w) dV(w)$$

is bounded from $L^{p,q}_\nu(T_\Omega)$ to $L^{p,q}_{\nu+mq}(T_\Omega)$ if and only if the following conditions are satisfied:

$$\max\{-mq + \frac{n}{r} - 1, q(-m + \frac{n}{r} - 1) - \frac{n}{r} + 1\} < \nu < \min\{q(\nu - \frac{n}{r} + 1) + \frac{n}{r} - 1, q\nu - \frac{n}{r} + 1\}.$$

Proof. The operator K defined by $K(f)(z) = \Delta^{-m}(\Im z) f(z)$ is an isometric isomorphism of $L^{p,q}_\nu(T_\Omega)$ to $L^{p,q}_{\nu+mq}(T_\Omega)$. Since for every f in $L^{p,q}_\nu(T_\Omega)$,

$$Q^+ f(z) = \int_{T_\Omega} |B_{\nu+m}(z,w)| \Delta^{-m}(\Im w) f(w) \Delta^{\nu+m-\frac{n}{r}}(\Im w) dV(w) = T^+_{0,\nu+m-\frac{n}{r},\nu+m}(Kf)(z),$$

the corollary follows from Theorem 6.3.3. □

Remark 6.3.8. The above corollary in the case $r = 2$ is Proposition 3.5 of [10].

We recall that $P^+_\mu = T^+_{0,\mu-\frac{n}{r},\mu}$. The boundedness of P^+_μ has been obtained in [14] for the case of the light cone. The following corollary is its generalization.

Corollary 6.3.9. *Let $\mu, \nu \in \mathbb{R}$ and $1 \leq p, q < \infty$. Then P^+_μ is bounded in $L^{p,q}_\nu(T_\Omega)$ if and only if $\mu, \nu > \frac{n}{r} - 1$ and*

$$\max\{\frac{\nu - (\frac{n}{r} - 1)}{\mu - (\frac{n}{r} - 1)}, \frac{\nu + \frac{n}{r} - 1}{\mu}\} < q < \frac{\nu + \frac{n}{r} - 1}{\frac{n}{r} - 1}.$$

Recall that the Bergman projection P_μ is defined for $f \in L^2_\mu(T_\Omega)$ by

$$P_\mu f(z) = \int_{T_\Omega} B_\mu(z,w) f(w) \Delta^{\mu - n/r}(\Im w) dV(w),$$

where the Bergman kernel B_μ is given by (6.1.3). $P_\mu f(z)$ defines a holomorphic function in T_Ω whenever the above integral is absolutely convergent. This is also the case if we consider $P_\mu f(z)$ with $f \in L^{p,q}_\nu(T_\Omega)$. Using the notation $\tilde{q}_{\nu,p} = \frac{\nu + \frac{n}{r} - 1}{(\frac{n}{rp'} - 1)_+}$ with $\tilde{q}_{\nu,p} = \infty$ if $n/r \leq p'$, we have the following proposition (see also Lemma 4.23 in [13] for the case $\mu = \nu$).

Proposition 6.3.10. *Let $\mu, \nu \in \mathbb{R}$ and $1 \leq p, q < \infty$. If P_μ extends to a bounded operator on $L^{p,q}_\nu(T_\Omega)$, then $B_\mu(z, i\underline{\mathbf{e}}) \in L^{p,q}_\nu$ and $\Delta^{\mu-\nu}(\Im z) B_\mu(z, i\underline{\mathbf{e}}) \in L^{p',q'}_\nu$. The latter is equivalent to the following conditions: $\nu > \frac{n}{r} - 1$ and $p(\frac{n}{r} - 1 - \mu) < 2\frac{n}{r} - 1 < p(\frac{n}{r} + \mu)$,*

$$\max\{\frac{\nu - \frac{n}{r} + 1}{(\mu - \frac{n}{r} + 1)_+}, \frac{\nu + \frac{n}{r} - 1}{(\mu + \frac{n}{rp'})_+}\} < q < \tilde{q}_{\nu,p}.$$

Proof. Let P_μ^* be the adjoint operator of P_μ with respect to the pairing $<,>_\nu$. We have

$$P_\mu^* f(z) = \Delta^{\mu-\nu}(\Im z)\int_{T_\Omega} B_\mu(z,w)f(w)\Delta^{\nu-n/r}(\Im w)dV(w), \quad f \in L_\nu^{p',q'}.$$

Testing P_μ with $f_1(z) = \chi_{B_1(i\underline{\mathbf{e}})}(z)\Delta^{-\mu+\frac{n}{r}}(\Im z)$ and P_μ^* with $f_2(z) = \chi_{B_1(i\underline{\mathbf{e}})}(z)\Delta^{-\nu+\frac{n}{r}}(\Im z)$ where $B_1(i\underline{\mathbf{e}})$ is the Euclidean ball of radius 1 centered at $i\underline{\mathbf{e}}$, it follows from the mean value property that $P_\mu f_1(z) = CB_\mu(z,i\underline{\mathbf{e}})$ and $P_\mu^* f_2(z) = C\Delta^{\mu-\nu}(\Im z)B_\mu(z,i\underline{\mathbf{e}})$. Consequently, we have $B_\mu(z,i\underline{\mathbf{e}}) \in L_\nu^{p,q}$ and $\Delta^{\mu-\nu}(\Im z)B_\mu(z,i\underline{\mathbf{e}}) \in L_\nu^{p',q'}$. Thus, by Lemma 6.3.2 this is equivalent to $\nu + (\mu-\nu)q' > \frac{n}{r} - 1$, $\nu > \frac{n}{r} - 1$, $\mu + \frac{n}{r} > (2\frac{n}{r} - 1)\max(\frac{1}{p'}, \frac{1}{p})$ and $\mu + \frac{n}{r} > \max\{\frac{n}{rp'} + \frac{\nu+(\mu-\nu)q'+\frac{n}{r}-1}{q'}, \frac{n}{rp} + \frac{\nu+\frac{n}{r}-1}{q}\}$. That is, $\nu > \frac{n}{r} - 1$, $\mu + \frac{n}{r} > (2\frac{n}{r} - 1)\max(\frac{1}{p'}, \frac{1}{p})$, and $\max\{\frac{\nu-\frac{n}{r}+1}{(\mu-\frac{n}{r}+1)_+}, \frac{\nu+\frac{n}{r}-1}{(\mu+\frac{n}{rp'})_+}\} < q < \tilde{q}_{\nu,p}$. □

Theorem 6.3.11. *The operator T^+ is bounded on $L^\infty(T_\Omega)$ if and only if $\alpha > \frac{n}{r} - 1$, $\beta > -1$ and $\gamma = \alpha + \beta + \frac{n}{r}$.*

Proof. We first prove the sufficiency. For any $f \in L^\infty(T_\Omega)$, we have

$$\begin{aligned}
|T^+f(x+iy)| &\leq \Delta^\alpha(y)\int_{T_\Omega} |B_\gamma(x+iy,u+iv)||f(u+iv)|\Delta^\beta(v)dudv \\
&\leq ||f||_\infty \Delta^\alpha(y)\int_{T_\Omega} |\Delta^{-(\gamma+\frac{n}{r})}(\frac{x-u+i(y+v)}{i})|\Delta^{(\beta+\frac{n}{r})-\frac{n}{r}}(v)dudv \\
&\leq C||f||_\infty \Delta^{\alpha-\gamma+\beta+\frac{n}{r}}(y) \\
&= C||f||_\infty,
\end{aligned}$$

where the third inequality follows from Lemma 6.3.2 under the hypotheses.

We now prove the necessary condition. First, we show that if T^+ is bounded on $L^\infty(T_\Omega)$, then the equality $\gamma = \alpha + \beta + \frac{n}{r}$ holds. For $f \in L^\infty(T_\Omega)$, we define the function f_R, $R > 0$, by $f_R(z) = f(Rz)$ for any $z \in T_\Omega$. The function f_R belongs to $L^\infty(T_\Omega)$ and we have $||f_R||_\infty \leq ||f||_\infty$. Using the homogeneity of the determinant function, we obtain

$$||T^+(f_R)||_\infty = R^{r(\gamma+\frac{n}{r})-r\alpha-r\beta-2n}||T^+f||_\infty.$$

It follows from the hypotheses that there exists a positive constant C such that $||T^+(f_R)||_\infty \leq C||f_R||_\infty$. This implies that $R^{r(\gamma+\frac{n}{r})-r\alpha-r\beta-2n}||T^+f||_\infty \leq C||f||_\infty$ for all $R > 0$, which necessarily implies that $\gamma = \alpha+\beta+\frac{n}{r}$. Now, we test T^+ on the function $f(x+iy) = \chi_{|x|<1}g(y)$, where g is a positive function compactly supported on the intersection of the cone with the Euclidean ball of radius δ centered at 0. From Lemma 6.3.1, we have that for x, y with $|x| < \frac{1}{4}$, $|y| < \delta$, the following inequality holds:

$$\Delta^\alpha(y)\int_{v\in\Omega,|v|<\delta} \Delta^{-\gamma}(y+v)g(v)\Delta^\beta(v)dv \leq CT^+f(x+iy) \leq C||f||_\infty.$$

We already know that by homogeneity of the kernel, we can replace δ by an arbitrary positive constant K. Thus by density of compactly supported functions, we can just write the left hand side of the above inequality without any bound on the integral. Taking $g(v) = 1$, it follows that we should have

$$\Delta^{\alpha}(y) \int_{\Omega} \Delta^{-\gamma}(y+v)\Delta^{\beta}(v)dv < \infty.$$

It follows easily from Proposition 5.3.4 that we should have $\beta > -1$ and $-\gamma+\beta < -2\frac{n}{r}+1$. Thus, using the equality previously obtained, we deduce that $\alpha > \frac{n}{r} - 1$. This completes the proof of the theorem. □

Although the conditions for the boundedness of T^+ are generally only sufficient for the boundedness of T, in the case of $L^\infty(T_\Omega)$ they are also necessary, as we show in the next result.

Theorem 6.3.12. *The operator T is bounded on $L^\infty(T_\Omega)$ if and only if $\alpha > \frac{n}{r}-1$, $\beta > -1$ and $\gamma = \alpha + \beta + \frac{n}{r}$.*

Proof. We only have to prove necessity. Let T be bounded on $L^\infty(T_\Omega)$. The condition $\gamma = \alpha+\beta+\frac{n}{r}$ follows on the same way as in the proof of the previous theorem. Let $w = \xi + it \in T_\Omega$ be fixed and consider the function f_w given by $f_w(x+iy) = \frac{|B_\gamma(\xi+it,x+iy)|}{B_\gamma(\xi+it,x+iy)}\chi_{|x|<1}g(y)$; where g is a positive function compactly supported on the intersection of the cone with the Euclidean ball of radius δ centered at 0. Testing T with f_w and taking $x + iy = w$, we obtain with the same reasoning as in the proof of Theorem 6.3.3 that we have $\beta > -1$, $-\gamma + \beta < -2\frac{n}{r} + 1$ and consequently $\alpha > \frac{n}{r} - 1$. □

6.4 The topological dual of $A_\nu^{p,q}(T_\Omega)$, $1 < q < q_\nu$

We recall the following notations:

$$\tilde{q}_{\nu,p} = \frac{\nu + \frac{n}{r} - 1}{(\frac{n}{rp'} - 1)_+}, \qquad q_{\nu,p} = \min\{p, p'\}q_\nu, \quad \text{and} \quad q_\nu = 1 + \frac{\nu}{\frac{n}{r} - 1}$$

with $\tilde{q}_{\nu,p} = \infty$ if $n/r \leq p'$. It is clear that $1 < q_\nu < q_{\nu,p} < \tilde{q}_{\nu,p}$. By density of the intersection $A_\nu^{p,q} \cap A_\mu^2$ in $A_\nu^{p,q}$ (see [12]), we have the following reproducing formula for all $\alpha > \frac{n}{r} - 1$ and $f \in A_\nu^{p,q}$ with $1 \leq p < \infty$ and $1 \leq q < \tilde{q}_{\nu,p}$:

$$f(z) = \int_{T_\Omega} B_\alpha(z,w)f(w)\Delta^{\alpha-\frac{n}{r}}(\Im w)dV(w), \quad z \in T_\Omega. \tag{6.4.1}$$

Remark 6.4.1. In fact, formula (6.4.1) holds for all $f \in A^2_\mu$ and then by density for all $f \in A^{p,q}_\nu$. We will give more comments about this in the next chapter.

The following theorem characterizes the topological dual space of the Bergman space $A^{p,q}_\nu(T_\Omega)$ for some values of p, q and ν for which the Bergman projection is not necessarily bounded.

Theorem 6.4.2. *Let $\nu > \frac{n}{r} - 1$ be real, $1 < p < \infty$ and $1 < q < q_\nu$. If μ is a sufficiently large real number so that $\mu > \frac{n}{r} - 1$ and $1 < q' < q_\mu$, then the topological dual space $(A^{p,q}_\nu)^*$ of the Bergman space $A^{p,q}_\nu$ identifies with $A^{p',q'}_\mu$ under the integral pairing*

$$\langle f, g\rangle_\alpha = \int_{T_\Omega} f(w)\overline{g(w)}\Delta^{\alpha-\frac{n}{r}}(\Im w)dV(w),$$

where $\alpha = \frac{\nu}{q} + \frac{\mu}{q'}$, $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$.

Proof. We have the equality

$$\int_{T_\Omega} f(z)\overline{g}(z)\Delta^{\alpha-\frac{n}{r}}(\Im z)dV(z) = \int_{T_\Omega} (\Delta^{\frac{\nu-\frac{n}{r}}{q}}(\Im z)f(z))(\Delta^{\frac{\mu-\frac{n}{r}}{q'}}(\Im z)\overline{g}(z))dV(z).$$

Since for every $f \in A^{p,q}_\nu$, the function $\Delta^{\frac{\nu-\frac{n}{r}}{q}}(\Im z)f(z)$ is in $L^{p,q}(T_\Omega, dz)$ and for every $g \in A^{p',q'}_\mu$, the function $\Delta^{\frac{\mu-\frac{n}{r}}{q'}}(\Im z)g(z)$ is in $L^{p',q'}(T_\Omega, dz)$, it follows that the given form is well-defined and every $g \in A^{p',q'}_\mu$ defines an element of $(A^{p,q}_\nu)^*$ given by the above integral pairing. The injectivity of the mapping $g \in A^{p',q'}_\mu \mapsto \langle ., g\rangle_\alpha$ follows by testing with $f = B_\alpha(., w)$. Indeed, $f = B_\alpha(., w)$ belongs to $A^{p,q}_\nu$ by Lemma 6.3.2 since $\alpha > \frac{n}{r} - 1$ and $q > q'_\mu > \frac{\mu+\frac{n}{r}-1}{\mu+\frac{n}{rp'}}$. Now using the reproducing formula (6.4.1) we obtain that if $g \in A^{p',q'}_\mu$ is such that $\langle h, g\rangle = 0$ for all $h \in A^{p,q}_\nu$, then in particular $0 = \langle B_\alpha(., w), g\rangle = \overline{g(w)}$ for all $w \in \mathcal{D}$ and so $g \equiv 0$.

Now let us show that every element $\mathcal{M}$ of $(A^{p,q}_\nu)^*$ can be represented by an element g of $A^{p',q'}_\mu$. By the Hahn-Banach theorem, there exists a function $h \in L^{p',q'}_\nu$ satisfying $||h||_{L^{p',q'}_\nu} = ||\mathcal{M}||$ such that for any $f \in A^{p,q}_\nu$,

$$\mathcal{M}(f) = \int_{T_\Omega} F(z)\overline{h(z)}\Delta^{\nu-\frac{n}{r}}(\Im z)dV(z).$$

Let us set $k(z) = \Delta^{\frac{\nu-\mu}{q'}}(\Im z)h(z)$. Then $k \in L^{p',q'}_\mu$ and we have

$$\int_{T_\Omega} f(z)\overline{h(z)}\Delta^{\nu-\frac{n}{r}}(\Im z)dV(z) = \int_{T_\Omega} f(z)\overline{k(z)}\Delta^{\alpha-\frac{n}{r}}(\Im z)dV(z).$$

It is easy to see that $\mu < \min\{q'(\alpha - \frac{n}{r} + 1) + \frac{n}{r} - 1, q'\alpha - \frac{n}{r} + 1\}$ and $\nu < \min\{q(\alpha - \frac{n}{r} + 1) + \frac{n}{r} - 1, q\alpha - \frac{n}{r} + 1\}$. Thus, P_α is bounded on $L^{p',q'}_\mu$ and on $L^{p,q}_\nu$. If we set $g = P_\alpha(k)$, g belongs to $A^{p',q'}_\mu$ and we clearly have

$$\mathcal{M}(f) = \langle f, k\rangle_\alpha = \langle P_\alpha f, k\rangle_\alpha = \langle f, P_\alpha k\rangle_\alpha = \langle f, g\rangle_\alpha.$$

We have used the fact that since P_α is bounded on $L^{p,q}_\nu$, it reproduces functions of $A^{p,q}_\nu$. The proof is complete. $\square$

Chapter 7

Analytic Besov spaces on tube domains

We give various equivalent formulations to the (partially) open problem about L^p-boundedness of Bergman projections in tubes over cones. Namely, we show that such boundedness is equivalent to the duality identity between Bergman spaces, $A^{p'} = (A^p)^*$, and also to a Hardy type inequality related to the wave operator. We introduce analytic Besov spaces in tubes over cones, for which such Hardy inequalities play an important role. For $p \geq 2$ we identify as a Besov space the range of the Bergman projection acting on L^p, and also the dual of $A^{p'}$. For the Bloch space $\mathbb{B}^\infty$ we give in addition new necessary conditions on the number of derivatives required in its definition.

7.1 Introduction

Let T_Ω be a symmetric domain of tube type in $\mathbb{C}^n$, that is $T_\Omega = \mathbb{R}^n + i\Omega$, where Ω is an *irreducible symmetric cone* in $\mathbb{R}^n$. We still write r for the rank of Ω and $\Delta(x)$ for the associated determinant function as in the previous chapter. We shall denote by $\mathcal{H}(T_\Omega)$ the space of holomorphic functions on T_Ω.

A major open question in these domains concerns the L^p boundedness of the *Bergman projection* [9, 13, 14]. Let $A^p_\nu(T_\Omega)$ denote the subspace of holomorphic functions in $L^p_\nu = L^p(T_\Omega, \Delta(y)^{\nu-n/r} dx\, dy)$. These spaces are nontrivial (i.e. $A^p_\nu \neq \{0\}$) only if $\nu > \frac{n}{r} - 1$ (see [38]). The usual (unweighted) Bergman spaces $A^p(T_\Omega)$ correspond to $\nu = \frac{n}{r}$. Let P_ν be the orthogonal projection mapping $L^2_\nu(T_\Omega)$ into $A^2_\nu(T_\Omega)$.

Conjecture 2. *Let $\nu > \frac{n}{r} - 1$. Then the Bergman projection P_ν admits a bounded extension to $L^p_\nu(T_\Omega)$ if and only if*

$$p'_\nu < p < p_\nu := \frac{\nu + \frac{2n}{r} - 1}{\frac{n}{r} - 1} - \frac{(1-\nu)_+}{\frac{n}{r} - 1}.$$

The necessity of the condition above was proved in [13]. The conjecture concerns the sufficiency. Note that the summand involving $(1-\nu)_+$ in the second term may only occur in the three dimensional forward light-cone ($n = 3$ and $r = 2$), where ν is allowed to take values below 1.

The problem of L^p-continuity of the Bergman projection has been studied in the papers [9, 13, 14], and completely settled for large ν in the case of light cones in [13]. Let us note

$$\tilde{p}_\nu := \frac{\nu + \frac{2n}{r} - 1}{\frac{n}{r} - 1}.$$

Then the necessary condition $p < \tilde{p}_\nu$ is given by the fact that by duality, the Bergman kernel has to belong to the dual space $L^{p'}_\nu(T_\Omega)$ (see also Proposition 6.3.10). As far as sufficient conditions are concerned, we refer to [52, 53] for the best sufficient conditions that are known, up to now, in the case of light cones. In general, it is proved in [13, 14] that the Bergman projection P_ν is bounded in $L^p_\nu(T_\Omega)$ for

$$\bar{p}'_\nu < p < \bar{p}_\nu := \frac{\nu + \frac{2n}{r} - 2}{\frac{n}{r} - 1}.$$

Let $\Box = \Delta(\frac{1}{i}\frac{\partial}{\partial x})$ denote the differential operator of degree r defined by the equality:

$$\Box\,[e^{i(x|\xi)}] = \Delta(\xi) e^{i(x|\xi)}, \quad \xi \in \mathbb{R}^n. \tag{7.1.1}$$

In cones of rank 1 and 2 this corresponds to $-i\partial_x$ (when T_Ω is the upper-half-plane) and $-(\partial^2_{x_1} - \partial^2_{x_2} - \ldots - \partial^2_{x_n})/4$ (when T_Ω is the forward light cone), which justifies the name of "wave operator" given to Δ. We denote by $\Box_z$ the extension of the operator $\Box$ to $\mathbb{C}^n$ given by $\Box_z = \Delta(\frac{1}{i}\frac{\partial}{\partial x})$. When there is no ambiguity, we write $\Box$ instead of $\Box_z$.

Let us first recall the following result of [14] which is a consequence of the mean value inequality for holomorphic functions.

Lemma 7.1.1. *Let $0 < p \leq \infty$ and $\nu \in \mathbb{R}$. Then*

$$\left\|\Delta(\Im m\,\cdot)\Box F\right\|_{L^p_\nu} \leq C \left\|F\right\|_{L^p_\nu} \tag{7.1.2}$$

for $F \in \mathcal{H}(T_\Omega)$.

In this chapter, we will be concerned with equivalent formulations of Conjecture 2 and some consequences in the formulation of the theory of analytic Besov spaces in these settings. We first prove that for $p \geq 2$, the validity of the reverse inequality of (7.1.2) is equivalent to the boundedness of P_ν on L^p_ν when $\nu > \frac{n}{r} - 1$. Clearly, we prove the following result.

Theorem 7.1.2. *Let $\nu > \frac{n}{r} - 1$. Then, for $p \geq 2$, the Bergman projection P_ν admits a bounded extension to $L^p_\nu(T_\Omega)$ if and only if there exists a constant C such that, for all $F \in A^p_\nu$ we have*

$$\iint_{T_\Omega} |F(x+iy)|^p \, \Delta^{\nu - \frac{n}{r}}(y)\, dx\, dy \;\leq\; C \iint_{T_\Omega} \left|\Delta(y) \Box F(x+iy)\right|^p \Delta^{\nu - \frac{n}{r}}(y)\, dx\, dy. \quad (7.1.3)$$

Such an inequality is called *Hardy Inequality* by reference to the one dimensional case. More comments on Hardy inequalities for holomorphic functions have been done in [22], where a weaker statement has been announced.

Remark 7.1.3.

- We remark that (7.1.3) is always valid when $1 \leq p \leq 2$, as it can be proved, for instance, from an explicit formula for F in terms of $\Box F$ involving the fundamental solution of the Box operator (see [22]). However, in this range (7.1.3) has no implications in terms of boundedness of Bergman projections.
- Let us mention that a weak form of Theorem 7.1.2 for forward light cones had also already been given in [14], where it was the key argument for proving the continuity of the Bergman projection.
- We will refer to (7.1.3) as Hardy's inequality for the parameters (p, ν).

We will prove Theorem 7.1.2 in Section 3, and add more comments on Hardy inequalities.

The second equivalent formulation of Conjecture 2 concerns duality.

Theorem 7.1.4. *Let $\nu > \frac{n}{r} - 1$ and $1 < p < \infty$. Then P_ν admits a bounded extension to $L^p_\nu(T_\Omega)$ if and only if the natural mapping of $A^{p'}_\nu$ into $(A^p_\nu)^*$ is an isomorphism.*

Remark 7.1.5. If $p > \tilde{p}_\nu'$, then the inclusion $\Phi : A^{p'}_\nu \hookrightarrow (A^p_\nu)^*$ is injective (see the proof of duality result in the previous chapter), and hence boundedness of P_ν is actually equivalent to surjectivity of Φ. When $p \geq \tilde{p}_\nu$ these two properties fail, and $(A^p_\nu)^*$ is a space strictly larger than $A^{p'}_\nu$ which we do not know how to identify.

The two theorems above give two equivalent formulations of the boundedness of the Bergman projection for $p > 2$. When $1 \leq p < 2$ is such that the projection P_ν is not bounded, then we can still describe the dual space of A^p_ν in terms of equivalence classes of holomorphic functions, and more precisely in terms of Besov spaces that we study in Section 4. Equivalence classes appear naturally in this setting since the injectivity of Φ (or equivalently of $\Box|_{A^{p'}_\nu}$) fails when $p < \tilde{p}'_\nu$. We define analytic Besov spaces $\mathbb{B}^p_\nu$, for $\nu \in \mathbb{R}$ and $1 \leq p < \infty$, by

$$\mathbb{B}^p_\nu := \{F \ : \ \Delta^m(\Im \cdot)\Box^m F \in L^p_\nu\}$$

for m large enough. The smallest possible value for m in the above definition is related to the validity of the Hardy inequality for some other weight, and one has to deal with equivalence classes modulo holomorphic functions that are annihilated by powers of the Box operator when m cannot be taken equal to 0. For the one dimensional case and bounded symmetric domains, we refer to [47, 115, 116]. Here, compared to the case of bounded symmetric domains, it is more difficult to deal with equivalence classes.

Let us mention the following special family of Besov spaces corresponding to the weight $\nu = -n/r$ in the above definition that is,

$$\mathbb{B}^p \ = \ \left\{ F \in \mathcal{H}(T_\Omega) \ : \ \Delta^m(\Im \cdot)\Box^m F \in L^p(d\lambda) \right\}.$$

Here $d\lambda = \Delta^{-\frac{2n}{r}}(y)dx\,dy$ denotes the invariant measure under conformal transformations of T_Ω. These are the analog for T_Ω of the Besov spaces introduced by Arazy and Yan in bounded symmetric domains [1, 109, 110]. The space $\mathbb{B}^p$ is the right range of symbols of Hankel operators in the Schatten class $\mathcal{S}_p$ [24, 115] (see also the next chapter). For $p = \infty$, the Besov space is known as the Bloch space (see e.g. [7, 8]).

Among our results we shall prove the following. Let $P^{(k)}_\nu(f)$ denotes the equivalence class $P_\nu(f) + \ker \Box^k$ (defined at least for f in the dense set $L^2_\nu \cap L^p_\mu$).

Theorem 7.1.6. *Let $\nu > \frac{n}{r} - 1$, $2 \leq p \leq \infty$ and $k \geq k_0(p, \nu)$. Then*

1.- For every real $\mu \leq \nu$, the operator $P^{(k)}_\nu$ extends continuously from L^p_μ onto $\mathbb{B}^p_\mu$.

2.- The dual space $(A^{p'}_\nu)^*$ identifies with $\mathbb{B}^p_\nu$, under the pairing

$$\langle F, G\rangle_{\nu,k} = \int_{T_\Omega} F(z)\,\Delta^k(\Im m\,z)\,\overline{\Box^k G(z)}\,dV_\nu(z), \quad F \in A^{p'}_\nu, \quad G \in \mathbb{B}^p_\nu.$$

Boundary values of Besov spaces can be defined via a Littlewood-Paley decomposition of the cone Ω, seen as supporting the function g, if the Laplace transform of g is the function under consideration. This allows to choose a representative holomorphic function in the equivalence class for much smaller values of ν. In order to do this, we shall go back to the definition of Besov spaces as Fourier-Laplace transforms of the Besov spaces at the boundary, as in [13].

7.2 Bergman kernels and reproduction formulas

7.2.1 Some prerequisites

Below we shall use some invariance properties of determinants and Box operators. To introduce them we need to recall some basic facts about symmetric cones introduced in Chapter 5.

Considering $V = \mathbb{R}^n$ as a Jordan algebra, we still denote its unit element by $\mathbf{e}$ (think of the identity matrix in the cone of positive definite symmetric matrices, or the point $\mathbf{e} = (1, \mathbf{0})$ in the forward light cone). We recall that G is the identity component of the group of invertible linear transformations which leave the cone Ω invariant. We have already seen that G acts transitively on Ω. The determinant function is also preserved by G, in such a way that

$$\Delta(gy) = \Delta(g\mathbf{e})\Delta(y) = \mathrm{Det}(g)^{\frac{r}{n}}\Delta(y), \quad \forall\, g \in G,\ y \in \Omega. \tag{7.2.1}$$

It follows from this formula that an invariant measure in Ω is given by $\Delta(y)^{-\frac{n}{r}}\,dy$. More precisely, we have the following result which follows by a change of variable and formula (7.2.1) (see [12]).

Proposition 7.2.1. *Let Ω be a symmetric cone. Consider in Ω the measure :*

$$\mu(E) = \int_E \frac{dy}{\Delta(y)^{\frac{n}{r}}}, \quad E \subset \Omega.$$

Then μ is G-invariant, i.e., $\mu(g \cdot E) = \mu(E)$ for all $g \in G$.

The invariance of the Box operator under the action of G is an easy consequence of its definition and the invariance of the determinant function (see [14]), namely

$$\Box\big[F(g\cdot)\big] \;=\; \Delta(g\mathbf{e})\,\big[\Box F\big](g\cdot) \;=\; \mathrm{Det}(g)^{\frac{r}{n}}\big[\Box F\big](g\cdot), \quad \forall\, g \in G. \tag{7.2.2}$$

Another fundamental property is the following [40, p. 125]: for every $\alpha \in \mathbb{R}$ one has the identity in Ω

$$\Box\Delta^{\alpha} = b(\alpha)\Delta^{\alpha-1} \tag{7.2.3}$$

where

$$b(\alpha) = \alpha(\alpha + \frac{d}{2}) \cdots (\alpha + (r-1)\frac{d}{2}), \quad \frac{d}{2} = \frac{\frac{n}{r} - 1}{r - 1}.$$

The polynomial b is called the Bernstein polynomial of the determinant.

Remark 7.2.2. The polynomial b vanishes for the r values $0, \alpha_0, \cdots (r-1)\alpha_0$, where $\alpha_0 = -\frac{\frac{n}{r}-1}{r-1}$. Consequently, we have for example,

$$\Box \Delta^{-\frac{n}{r}+1}(y) = 0, \quad y \in \Omega. \tag{7.2.4}$$

7.2.2 Bergman kernels and Determinant function

Recall that the (weighted) Bergman projection P_ν is defined by

$$P_\nu F(z) = \int_{T_\Omega} B_\nu(z, w) F(w) dV_\nu(w),$$

where $B_\nu(z, w) = c_\nu \, \Delta^{-(\nu + \frac{n}{r})}((z - \overline{w})/i)$ is the reproducing kernel of A^2_ν, called Bergman kernel (see [40]). For simplicity, we have written $dV_\nu(w) := \Delta^{\nu - \frac{n}{r}}(v) du \, dv$, where $w = u + iv$ is an element of T_Ω. Observe from (7.2.3) that

$$\Box_z^m \left[B_\nu(z - \bar{w})\right] \, = \, c_{\nu,m} \, B_{\nu+m}(z - \bar{w}) \tag{7.2.5}$$

for a suitable constant $c_{\nu,m}$, and all $m \in \mathbb{N}$. We will need the integrability properties of the determinants and Bergman kernels provided by Proposition 5.3.4 and Lemma 6.3.2.

Remark 7.2.3. Lemma 6.3.2 means in particular, using (7.2.4), that for $p > \tilde{p}_\nu$ the function $F(z) = \Delta^{-\frac{n}{r}+1}(z + i\mathbf{e}) \in A^p_\nu$ and is annihilated by $\Box$; so, there is no Hardy inequality for such values of p. In this range of p, as mentioned in the introduction, the Bergman projection P_ν is not bounded in L^p_ν, so we have proved easily Theorem 7.1.2 for $p > \tilde{p}_\nu$. We shall concentrate on the other values of p later on.

Let us now recall the following density properties (see eg [14, 51]).

Lemma 7.2.4. *Let $1 \leq p < \infty$ and $\nu > \frac{n}{r} - 1$. Then, for all $1 \leq q \leq \infty$ and $\mu > \frac{n}{r} - 1$, the subspace $A^p_\nu \cap A^q_\mu$ is dense in A^p_ν. Moreover, $A^\infty \cap A^q_\mu$ is dense in A^∞ for the weak*-(L^∞, L^1) topology.*

Proof: Let us consider the case $p = \infty$, which is the only new part. If $F \in A^\infty$ then by part (1) of Lemma 5.3.3 the functions $\Delta^{-\alpha}((\varepsilon z + i\mathbf{e})/i)F(z)$ are in $A^p_\mu \cap A^\infty$ for large values of α, and we clearly have the required property when ε tends to 0 by Lebesgue dominated convergence theorem.

□

7.2.3 Integral operators

For the characterizations of Besov spaces, we shall need some integral estimates involving Bergman kernel functions. We will heavily make use of the following particular case of integral operators T and T^+ studied in the previous chapter.

$$T_{\nu,\alpha}F(z) = \Delta^{\alpha}(\Im m\, z)\int_{T_\Omega} B_{\nu+\alpha}(z,w)F(w)dV_\nu(w), \tag{7.2.6}$$

and

$$T^+_{\nu,\alpha}F(z) = \Delta^{\alpha}(\Im m\, z)\int_{T_\Omega} |B_{\nu+\alpha}(z,w)|F(w)dV_\nu(w), \tag{7.2.7}$$

when these integrals make sense. Observe that $P_\nu = T_{\nu,0}$. We recall the corresponding boundedness conditions of $T^+_{\nu,\alpha}$ on $L^p_\mu(T_\Omega)$.

Proposition 7.2.5. *Let $\alpha, \nu, \mu \in \mathbb{R}$ and $1 \le p < \infty$. Then the following conditions are equivalent:*

(a) The operator $T^+_{\nu,\alpha}$ is well defined and bounded on $L^p_\mu(T_\Omega)$.

(b) The parameters satisfy $\nu + \alpha > \frac{n}{r} - 1$ and the inequalities

$$\nu p - \mu > (\tfrac{n}{r} - 1)\max\{1, p-1\}, \quad \alpha p + \mu > (\tfrac{n}{r} - 1)\max\{1, p-1\}.$$

In particular, when $\nu = \mu > \frac{n}{r} - 1$ and when $p > (\mu + \frac{n}{r} - 1)/\mu$, the condition is satisfied for α large enough.

Proposition 7.2.6. *Let $\alpha, \nu \in \mathbb{R}$, with $\nu > \frac{n}{r} - 1$. Then the operator $T_{\nu,\alpha}$ (resp. $T^+_{\nu,\alpha}$) is bounded in L^∞ if and only if $\alpha > \frac{n}{r} - 1$.*

7.2.4 Reproducing formulas

We will make an extensive use of the following "*integration by parts*".

Proposition 7.2.7. *For $\nu > \frac{n}{r} - 1$, $1 \le p \le \infty$ and $F \in A^p_\nu$, $G \in A^{p'}_\nu$, we have the formula*

$$\int_{T_\Omega} F(z)\overline{G}(z)dV_\nu(z) = c_{\nu,m}\int_{T_\Omega} F(z)\overline{\Box^m G}(z)\Delta^m(\Im m\, z)dV_\nu(z). \tag{7.2.8}$$

Proof. We only need to show the identity (7.2.8) for $p = 2$ since the general case follows by density.Using the reproducing formula (6.1.4), derivation under the integral, Lemma 6.3.2 and Fubini's theorem we easily obtain for any $G \in A^2_\nu$ and for all $z \in T_\Omega$,

$$\int_{T_\Omega} B_\nu(z,w)G(w)dV_\nu(w) = \int_{T_\Omega} B_{\nu+m}(z,w)G(w)\Delta^m(\Im m\, w)dV_\nu(w). \tag{7.2.9}$$

That is the formula holds for $G = B_\nu(., w)$ and all $F \in A^2_\nu$. Using the reproducing formula (6.1.4) Fubini's theorem and the identity (7.2.9), and setting

$$I(F,G) = \int_{T_\Omega} F(z)\overline{\Box^m G}(z)\Delta^m(\Im m\, z)dV_\nu(z),$$

we obtain

$$\begin{aligned} I(F,G) &= \int_{T_\Omega} F(z)\left(c_{\nu,m}\int_{T_\Omega} B_{\nu+m}(w,z)\overline{G(w)}dV_\nu(w)\right)\Delta^m(\Im m\, z)dV_\nu(z) \\ &= c_{\nu,m}\int_{T_\Omega}\overline{G(w)}\left(\int_{T_\Omega} B_{\nu+m}(w,z)F(z)\Delta^m(\Im m\, z)dV_\nu(z)\right)dV_\nu(w) \\ &= c_{\nu,m}\int_{T_\Omega} F(w)\overline{G}(w)dV_\nu(w). \end{aligned}$$

□

We can now write the following general reproducing formula. In the next proposition, we write c for some constant that depends on the parameters involved.

Proposition 7.2.8. *Let $\nu > \frac{n}{r} - 1$ and $1 \le p \le \infty$. For all $F \in A^p_\nu$ we have the formula*

$$\Box^\ell F(z) = c\int_{T_\Omega} B_{\nu+\ell}(z,w)\Box^m F(w)\Delta^m(\Im m\, w)dV_\nu(w) \tag{7.2.10}$$

for $m \ge 0$ and ℓ large enough such that $B_{\nu+\ell}(z,\cdot)$ is in $L^{p'}_\nu$. In particular, when $1 \le p < \tilde{p}_\nu$, the formula is valid with $\ell = 0$.

Proof: We can assume that $m = 0$. If not, we use (7.2.8). It is true for $p = 2$ and $\ell = 0$ because of the reproducing property of the Bergman projection. Derivation under the integral and (7.2.5) gives also the case $\ell > 0$. We then use density for the general case. □

Corollary 7.2.9. *Let $1 \le p < \tilde{p}_\nu$ and $\nu > \frac{n}{r} - 1$. Then every $F \in A^p_\nu$ can be written as*

$$F(z) = \int_{T_\Omega} B_\nu(z,w)F(w)dV_\nu(w). \tag{7.2.11}$$

We shall state two more results which can be similarly proved by density and absolute convergence of the involved integrals (together with Lemma 6.3.2 to verify the statements about the Bergman kernels).

Proposition 7.2.10. *Let $\nu > \frac{n}{r} - 1$ and $\alpha > \frac{n}{r} - 1$. Then $B_{\nu+\alpha}(\cdot, i\mathbf{e}) \in L^1_\nu$, and for all holomorphic F with $\Delta^\alpha(\Im m\, z)F(z) \in L^\infty$ and all $m \ge 0$ we have*

$$F(z) = c\int_{T_\Omega} B_{\nu+\alpha}(z,w)\,\Box^m F(w)\,\Delta^{\alpha+m}(\Im m\, w)\,dV_\nu(w). \tag{7.2.12}$$

Proposition 7.2.11. *Let $\mu, \nu, \alpha \in \mathbb{R}$ and $1 \le p < \infty$ satisfying*

$$\nu + \alpha > \tfrac{n}{r} - 1, \quad \nu p - \mu > (p-1)(\tfrac{n}{r} - 1) \quad \textit{and} \quad \mu + \alpha p > (p-1)(\tfrac{n}{r} - 1) - \tfrac{n}{r}.$$

Then, $\Delta^{\nu-\mu}(\Im m\, z) B_{\nu+\alpha}(z, i\mathbf{e}) \in L^{p'}_{\mu}$, and for all holomorphic F with $\Delta^{\alpha}(\Im m\, z)F(z) \in L^p_\mu$ we have

$$F(z) = \int_{T_\Omega} B_{\nu+\alpha}(z, w)\, F(w)\, \Delta^{\alpha}(\Im m\, w)\, dV_\nu(w). \tag{7.2.13}$$

7.3 Hardy-type inequality and duality

We consider in this section, the proofs of equivalent formulations of boundedness of Bergman projection.

7.3.1 Equivalence between boundedness and Hardy's inequality

We prove in this subsection the equivalence between the validity of the Hardy inequality (7.1.3) and the boundedness of the Bergman projection. Let us first prove the following lemma.

Lemma 7.3.1. *Let $\nu > \frac{n}{r} - 1$ and $2 \le p \le \tilde{p}_\nu$. Then,*

$$\left\|\Box F\right\|_{L^p_{\nu+p}} \le C \left\|\Box^{m+1} F\right\|_{L^p_{\nu+(m+1)p}}, \quad \forall\, F \in A^p_\nu,\ \forall\, m \ge 1. \tag{7.3.1}$$

Proof. Using (7.2.8) we can write

$$\Box F(z) = c \int_{T_\Omega} B_{\nu+p}(z, w) \Box^m \left(\Box F(w)\right) \Delta^m(\Im m\, w) dV_{\nu+p}(w),$$

since $\Box F \in A^p_{\nu+p}$ and $B_{\nu+p}(\cdot, z) \in A^{p'}_{\nu+p}$. So the inequality (7.3.1) follows from the fact that the projector $P_{\nu+p}$ is bounded on $L^p_{\nu+p}$ (since the condition on p implies $p < \bar{p}_{\nu+p}$). $\Box$

Theorem 7.3.2. *Let $\nu > \frac{n}{r} - 1$. Then for $p \ge 2$, the Bergman projection P_ν admits a bounded extension to $L^p_\nu(T_\Omega)$ if and only if there exists a constant C such that for all $F \in A^p_\nu$, we have*

$$\left\|F\right\|_{L^p_\nu} \le C \left\|\Delta(\Im .) \Box F\right\|_{L^p_\nu}. \tag{7.3.2}$$

Proof. Let us first assume that P_ν is bounded, which implies in particular that $p < \tilde{p}_\nu$, that is, $B_\nu(z, \cdot)$ is in $A^{p'}_\nu$. Then the formula

$$F(z) = c \int_{T_\Omega} B_\nu(z, w) \Box F(w) \Delta(\Im m\, w) dV_\nu(w)$$

implies that F is the projection of the function $\Box F(w)\Delta(\Im w) \in L^p_\nu$. Thus, it follows from the continuity of the projection that

$$\|F\|_{L^p_\nu} = c\|P_\nu(\Delta(\Im .)\Box F)\|_{L^p_\nu} \leq C\|\Delta(\Im .)\Box F\|_{L^p_\nu}.$$

Next, consider $2 < p < \infty$ and assume that the inequality (7.3.2) holds. We can restrict to the range $2 < p \leq \tilde{p}_\nu$, since for larger values $p > \tilde{p}_\nu$, as we have seen above, the Box operator is not injective in A^p_ν, and hence Hardy's inequality does not hold.

Our proof uses Hardy's inequality (7.3.2), not only for the Box operator, but for its power $\Box^m$ with m large enough. It follows from Lemma 7.3.1 that our assumption that Hardy's inequality (7.3.2) holds implies that for all $F \in A^p_\nu$ and all positive integer m, we have the inequality

$$\iint_{T_\Omega} |F(x+iy)|^p \,\Delta^{\nu-\frac{n}{r}}(y)\,dx\,dy \;\leq\; C\iint_{T_\Omega} \left|\Delta^m(y)\Box^m F(x+iy)\right|^p \Delta^{\nu-\frac{n}{r}}(y)\,dx\,dy. \tag{7.3.3}$$

We want to prove the existence of some constant C such that for $f \in L^p_\nu \cap L^2_\nu$, we have the inequality

$$\|P_\nu f\|_{A^p_\nu} \leq C\|f\|_{L^p_\nu}.$$

Consider such an f with $\|f\|_{L^p_\nu} = 1$. Call $F := P_\nu f$. By Fatou's Lemma, it is sufficient to prove that the functions $F_\varepsilon(z) := F(z+i\varepsilon\mathbf{e})$, which belong to A^p_ν, have uniformly bounded norms. So, using (7.3.3), it is sufficient to prove that $\Box^m F_\varepsilon$ is uniformly in $L^p_{\nu+pm}$ for some m, which is a consequence of the fact that $\Box^m F$ itself is in $L^p_{\nu+pm}$ for some m (see eg [51, Corol. 3.9]). To prove this, we use the identity

$$\Box^m F(z) = c\int_{T_\Omega} B_{\nu+m}(z,w)f(w)dV_\nu(w),$$

so that $\|\Box^m F\|_{L^p_{\nu+pm}} = c\,\|T_{\nu,m}f\|_{L^p_\nu}$, and if m is sufficient large we conclude from Theorem 6.3.3 This finishes the proof. $\Box$

7.3.2 Boundedness of Bergman projection and duality

We prove the following equivalence between the boundedness of the Bergman projection P_ν on $L^p_\nu(T_\Omega)$ and the isomorphism of the natural mapping of $A^{p'}_\nu$ into $(A^p_\nu)^*$.

Theorem 7.3.3. *Let $\nu > \frac{n}{r} - 1$ and $1 < p < \infty$. Then P_ν admits a bounded extension to $L^p_\nu(T_\Omega)$ if and only if the natural mapping of $A^{p'}_\nu$ into $(A^p_\nu)^*$ is an isomorphism.*

Proof. We first consider the case $\tilde{p}_\nu\,' < p < \infty$, for which the Bergman kernel $B_\nu(\cdot, w)$ belongs to A^p_ν. So, if F is in $A^{p'}_\nu$ and if the associated linear form $\Phi(F)$, given by

$$\langle \Phi(F), G\rangle_\nu = \int_{T_\Omega} G(z)\overline{F(z)}dV_\nu(z)$$

vanishes on A^p_ν, Corollary 7.2.9 implies that $F = 0$. Thus, $A^{p'}_\nu$ is embedded into the dual of A^p_ν. Assume that this embedding is onto, and hence by the closed graph theorem that it has a continuous inverse. Since every $f \in L^{p'}_\nu$ defines an element of $(A^p_\nu)^*$ by $G \mapsto \int_{T_\Omega} G(z)\overline{f(z)}dV_\nu(z)$, by assumption there exists $F \in A^{p'}_\nu$ such that

$$\int_{T_\Omega} G(z)\overline{f(z)}dV_\nu(z) = \int_{T_\Omega} G(z)\overline{F(z)}dV_\nu(z), \quad \forall\, G \in A^p_\nu$$

with $\|F\|_{A^{p'}_\nu} \leq c\|f\|_{L^{p'}_\nu}$. Taking for G the Bergman kernel, we see that F is the projection $P_\nu f$, so that P_ν maps L^p_ν continuously into itself.

Conversely, assume that P_ν is bounded in L^p_ν (and, by duality, on $L^{p'}_\nu$). Then we have the identity

$$\int_{T_\Omega} G(z)\overline{f(z)}dV_\nu(z) = \int_{T_\Omega} G(z)\overline{P_\nu f(z)}dV_\nu(z)$$

for all $f \in L^{p'}_\nu$ and $G \in A^p_\nu$. Indeed, use the fact that this equality is valid in L^2_ν, and density. Since every functional $\gamma \in (A^p_\nu)^*$ can be expressed by Hahn-Banach as $G \mapsto \langle G, f\rangle_\nu$ for some $f \in L^{p'}_\nu$ (with $\|f\|_{L^{p'}_\nu} = \|\gamma\|$), the above identity shows that the functional can be obtained from $P_\nu f \in A^{p'}_\nu$. So, under the assumption that P_ν is bounded in L^p_ν, the embedding $\Phi : A^{p'}_\nu \to (A^p_\nu)^*$ is an isomorphism.

It remains to consider the case when $1 \leq p \leq \tilde{p}_\nu'$, where we know that the Bergman projection is not bounded, and hence we want to show that Φ is not an isomorphism. First, it is easy to see that Φ is not injective when $1 \leq p < \tilde{p}_\nu'$. Indeed, in that range we may find a (non-null) function $F \in A^{p'}_\nu$ with $\Box F = 0$ (see Remark 7.2.3). Now, it follows from (7.2.8) that

$$\int_{T_\Omega} G(z)\overline{F}(z)dV_\nu(z) = c\int_{T_\Omega} G(z)\,\overline{\Box F(z)}\Delta(\Im m\, z)dV_\nu(z), \quad G \in A^p_\nu, \tag{7.3.4}$$

which implies $\Phi(F) \equiv 0$.

Let us now consider the end-point, $p = \tilde{p}_\nu'$. If F is in $A^{p'}_\nu$ then $\Box F$ is in $A^{p'}_{\nu+p'}$ and, by (7.3.4), the norm of $\Phi(F)$ is bounded by the norm of $\Box F$ in this space. So, if Φ was an isomorphism, we would have some constant C independent of F such that

$$\|F\|_{A^{p'}_\nu} \leq C\|\Box F\|_{A^{p'}_{\nu+p'}}.$$

This is exactly Hardy's inequality, which is not valid for $p' = \tilde{p}_\nu$, concluding the proof of the theorem. $\square$

The next corollary, which is implicitly contained in the previous proofs, will be used later on.

Corollary 7.3.4. *Let $\nu > \frac{n}{r} - 1$ and $1 \leq p < \tilde{p}_\nu$, and assume that the Hardy inequality* (7.1.3) *holds for (p, ν). Then for every positive integer m, the mapping $\Box^m : A^p_\nu \to A^p_{\nu+mp}$ is an isomorphism. In particular, for all $G \in A^p_{\nu+mp}$ the equation $\Box^m F = G$ has a unique solution in A^p_ν. Moreover,*

$$\|F\|_{A^p_\nu} \leq C \, \|G\|_{A^p_{\nu+mp}},$$

for some constant $C > 0$.

Proof. When $2 \leq p < \tilde{p}_\nu$, by the assumption and Lemma 7.3.1 we have the estimate $\|F\|_{A^p_\nu} \leq C \, \|\Box^m F\|_{A^p_{\nu+mp}}$, for all $F \in A^p_\nu$, so we only need to establish the surjectivity of $\Box^m$. Since by assumption and Theorem 7.3.2 the Bergman projection P_ν is bounded in L^p_ν, given any $G \in A^p_{\nu+mp}$, the function $F = P_\nu(\Delta^m(\Im m \cdot) G)$ belongs to A^p_ν. Moreover, by the reproducing formula (7.2.13) we have

$$\Box^m F(z) = \int_{T_\Omega} B_{\nu+m}(z, w) \, G(w) \, \Delta(\Im m \, w)^m \, dV_\nu(w) \; = \; c \, G(z),$$

which proves the surjectivity.

For $1 \leq p \leq 2$, injectivity follows directly from Proposition 7.2.8 (with $\ell = 0$). For surjectivity, we first remark that the conditions of Proposition 7.2.11 are satisfied (with $\nu = \mu$ and $\alpha = m$). Thus, formula (7.2.13) holds for all $G \in A^p_{\nu+mp}$, i.e.,

$$G(z) = \int_{T_\Omega} B_{\nu+m}(z, w) \, G(w) \, \Delta(\Im m \, w)^m \, dV_\nu(w).$$

For $G \in A^p_{\nu+mp}$ let

$$F(z) = \int_{T_\Omega} B_\nu(z, w) \, G(w) \, \Delta(\Im m \, w)^m \, dV_\nu(w).$$

Since $p' \geq 2$, F is well-defined and satisfies $\Box^m F = cG$. To conclude, it suffices to show that $F \in A^p_\nu$ or equivalently that

$$T_{-m, \nu+m-\frac{n}{r}, \nu} G(x + iy) = \Delta(y)^{-m} \int_{T_\Omega} B_\nu(z, w) \, G(w) \, \Delta(\Im m \, w)^m \, dV_\nu(w)$$

is in $L^p_{\nu+mp}$. This is an easy consequence of the boundedness of the operator $T_{-m, \nu+m-\frac{n}{r}, \nu}$ on $L^p_{\nu+mp}$ given by Theorem 6.3.3. The proof is complete. $\square$

Remark 7.3.5. An alternative proof of surjectivity in the case $1 \leq p \leq 2$ using the explicit formula involving the fundamental solution of $\Box$ can be found in [22, Prop. 3.1].

7.4 Besov spaces of holomorphic functions and duality

Throughout this section, given $m \in \mathbb{N}$, we shall denote

$$\mathcal{N}_m := \{F \in \mathcal{H}(T_\Omega) : \Box^m F = 0\}$$

and set

$$\mathcal{H}_m(T_\Omega) = \mathcal{H}(T_\Omega)/\mathcal{N}_m.$$

For simplicity, we use the following notation for the normalized Box operator: We write

$$\Delta^m \Box^m F(z) := \Delta^m(\Im m\, z)\Box^m F(z), \quad z \in T_\Omega. \tag{7.4.1}$$

For convenience, we shall use the same notations for holomorphic functions and for equivalence classes in $\mathcal{H}_m$. Remark that for $F \in \mathcal{H}_m(T_\Omega)$, we can speak of the function $\Box^m F$. Sometimes we shall write $\Box_z^{-m} G$ for the class in $\mathcal{H}_m(T_\Omega)$ of all $F \in \mathcal{H}(T_\Omega)$ with $\Box^m F = G$. When $G \in \mathcal{H}(T_\Omega)$ this class is non-empty by the standard theory of PDEs with constant coefficients (see eg [106]).

7.4.1 Definition of $\mathbb{B}_\mu^p(T_\Omega)$

Given $\mu \in \mathbb{R}$ and $1 \leq p < \infty$, we wish to define a Besov space $\mathbb{B}_\mu^p(T_\Omega)$ consisting of holomorphic F so that $\Delta^m \Box^m F \in L_\mu^p$ for sufficiently large m. The following proposition clarifies the dependence of such spaces on the parameter m.

Proposition 7.4.1. *Let $\mu \in \mathbb{R}$ and $1 \leq p < \infty$, and let $k, m \in \mathbb{Z}$, $0 \leq k \leq m$.*

(i) If $\Delta^k \Box^k F$ is in L_μ^p, then $\Delta^m \Box^m F$ is in L_μ^p and $\|\Delta^m \Box^m F\|_{L_\mu^p} \leq C \|\Delta^k \Box^k F\|_{L_\mu^p}$.

(ii) If $\mu + kp > \frac{n}{r} - 1$ and Hardy's inequality (7.1.3) holds for $(p, \nu = \mu + kp)$, then $\Delta^m \Box^m F \in L_\mu^p$ implies the existence of $\widetilde{F} \in \mathcal{H}(T_\Omega)$ so that $\Box^m \widetilde{F} = \Box^m F$ and $\|\Delta^k \Box^k \widetilde{F}\|_{L_\mu^p} \leq C \|\Delta^m \Box^m F\|_{L_\mu^p}$. Moreover the function $\widetilde{F}$ is uniquely determined modulo $\mathcal{N}_k$.

Proof. Assertion (i) follows from (7.1.2). We focus on assertion (ii). The assumption on Hardy's inequality implies that $\Box^{m-k} : A_{\mu+kp}^p \to A_{\mu+mp}^p$ is an isomorphism, by Proposition 7.3.4. Thus since $\Box^m F \in A_{\mu+mp}^p$, there is a unique $H \in A_{\mu+kp}^p$ with $\Box^{m-k} H = \Box^m F$. Now we take for $\widetilde{F}$ any holomorphic solution of $\Box^k \widetilde{F} = H$. $\Box$

Given $\mu \in \mathbb{R}$, $1 \leq p < \infty$ and $m \in \mathbb{N}$, we define the space

$$\mathbb{B}_\mu^{p,(m)} := \big\{ F \in \mathcal{H}_m(T_\Omega) \ : \ \Delta^m \Box^m F \in L^p_\mu \big\}$$

endowed with the norm $\|F\|_{\mathbb{B}^p_\mu} = \|\Delta^m \Box^m F\|_{L^p_\mu}$. Observe that each element of $\mathbb{B}_\mu^{p,(m)}$ is the equivalence class of all analytic solutions of the equation $\Box^m F = g$, for some $g \in A^p_{\mu+mp}$. Thus, the spaces are null when $\mu + mp \leq \frac{n}{r} - 1$. By the previous proposition, when $0 \leq k \leq m$ and $\mu + kp > \frac{n}{r} - 1$, the natural projection

$$\begin{array}{ccc} \mathbb{B}_\mu^{p,(k)} & \longrightarrow & \mathbb{B}_\mu^{p,(m)} \\ F + \mathcal{N}_k & \longmapsto & F + \mathcal{N}_m \end{array} \tag{7.4.2}$$

is an isomorphism of Banach spaces, provided Hardy's inequality (7.1.3) holds for the indices $(p, \nu = \mu + pk)$. This leads us to the following definition.

Definition 7.4.2. Given $\mu \in \mathbb{R}$ and $1 \leq p < \infty$, we define $\mathbb{B}^p_\mu := \mathbb{B}_\mu^{p,(k_0)}$ where $k_0 = k_0(p, \mu)$ is fixed by

$$k_0(p,\mu) := \min\big\{ k \geq 0 \ : \ \mu + kp > \tfrac{n}{r} - 1 \ \text{ and Hardy inequality holds for } (p, \mu + pk) \ \big\}. \tag{7.4.3}$$

Observe that $\mathbb{B}^p_\mu = A^p_\mu$ if and only if $k_0(p,\mu) = 0$. When $1 \leq p \leq 2$ we have $k_0(p,\mu) = \min\{k \geq 0 \ : \ \mu + kp > \frac{n}{r} - 1\}$. For $p > 2$, however, the exact value of $k_0(p,\mu)$ depends on Conjecture 2, and we only have the estimate

$$k_1(p,\mu) \leq k_0(p,\mu) \leq k_2(p,\mu)$$

where

$$\begin{aligned} k_1(p,\mu) &= \min\big\{ k \geq 0 \ : \ \mu + kp > \tfrac{n}{r} - 1 \quad \text{and} \quad p < p_{\mu+kp} \big\} \\ k_2(p,\mu) &= \min\big\{ k \geq 0 \ : \ \mu + kp > \tfrac{n}{r} - 1 \quad \text{and} \quad p < \bar{p}_{\mu+kp} \big\} \end{aligned}$$

A simple arithmetic manipulation shows that $k_1 \leq k_2 \leq k_1 + 1$, and hence $k_0 \in \{k_1, k_1 + 1\}$. Of course, the conjecture should be $k_0(p,\mu) = k_1(p,\mu)$, and hence we are at most one unit above the best possible integer in the definition of $\mathbb{B}^p_\mu$. Observe also that $k_1(p,\mu)$ and $k_2(p,\mu)$ can also be written as

$$k_1 = \min\Big\{ k \geq 0 \ : \ k + \tfrac{\mu}{p} > \max\big\{ (\tfrac{n}{r} - 1)\tfrac{1}{p}, \ (\tfrac{n}{r} - 1)(1 - \tfrac{2}{p}) - \tfrac{1}{p}, \ (\tfrac{n}{r} - 1)(\tfrac{1}{2} - \tfrac{1}{p}) \ \big\} \Big\},$$

$$k_2 = \min\Big\{ k \geq 0 \ : \ k + \tfrac{\mu}{p} > \max\big\{ (\tfrac{n}{r} - 1)\tfrac{1}{p}, \ (\tfrac{n}{r} - 1)(1 - \tfrac{2}{p}) \ \big\} \Big\}.$$

Thus, we have $k_0 = k_1 = k_2$ when $1 \leq p \leq 3$.

In all cases, we can summarize part of the discussion above in the following proposition.

Proposition 7.4.3. *Let $1 \le p < \infty$, $\mu \in \mathbb{R}$ and $k \ge k_0(p,\mu)$. Then*

$$\Box^k \colon \mathbb{B}^p_\mu \to A^p_{\mu+kp}$$

is an isomorphism of Banach spaces. In particular, $\mathbb{B}^p_\mu$ is an isomorphic copy of $A^p_{\mu+k_0p}$, and when $\mu > \frac{n}{r}-1$ then $\mathbb{B}^p_\mu = A^p_\mu$ for all $1 \le p < \bar{p}_\mu$.

Finally we define separately the special family

$$\mathbb{B}^p := \mathbb{B}^p_{-n/r} \;=\; \left\{ F \in \mathcal{H}(T_\Omega) \;:\; \Delta^k \Box^k F \in L^p(T_\Omega, d\lambda) \right\},$$

where k is sufficiently large and $d\lambda(z) = \Delta^{-\frac{2n}{r}}(\Im m\, z) dV(z)$, that is the invariant measure under conformal transformations of T_Ω. When $n = r = 1$, $\mathbb{B}^p$ is the analog in the upper half plane of the analytic Besov space studied by Arazy-Fisher-Peetre, Zhu and others [2, 3, 89, 114]. These spaces have also been considered in bounded symmetric domains by Yan (for $p = 2$), Arazy and Zhu [1, 110, 116].

7.4.2 Properties of $\mathbb{B}^p_\mu$: image of the Bergman operator and duality

Let $\nu > \frac{n}{r} - 1$, $1 \le p < \infty$ and $\mu \in \mathbb{R}$. When m is large we extend the definition of the Bergman projection P_ν to functions $f \in L^p_\mu$, by letting $P^{(m)}_\nu(f)$ be the equivalence class (in $\mathcal{H}_m$) of all holomorphic solutions of

$$\Box^m F \;=\; c_{\nu,m} \int_{T_\Omega} B_{\nu+m}(\cdot, w) f(w)\, dV_\nu(w).$$

The constant $c_{\nu,m}$ is as in (7.2.5), so that if $f \in L^2_\nu \cap L^p_\mu$ then $P^{(m)}_\nu(f) = P_\nu(f) + \mathcal{N}_m$, and in this sense we say that $P^{(m)}_\nu$ is an extension of the Bergman projection. Observe that $P^{(m)}_\nu$ is well defined and bounded from L^p_μ into $\mathbb{B}^{p,(m)}_\mu$ if and only if $T_{\nu,m}$ is bounded in L^p_μ, and in particular, by Lemma 7.2.5, when $p\nu - \mu > \max(1, p-1)(\frac{n}{r} - 1)$ and m is sufficiently large. Moreover, it follows from the reproducing formulas that the operator is onto. Indeed, by Proposition 7.2.11, every $F \in \mathbb{B}^{p,(m)}_\mu$ satisfies

$$\Box^m F(z) \;=\; \int_{T_\Omega} B_{\nu+m}(z, w)\, \Box^m F(w)\, \Delta(\Im m\, w)^m \, dV_\nu(w)$$

provided m is sufficiently large, from which it follows $F = cP^{(m)}_\nu(\Delta^m \Box^m F)$. Therefore we have shown the following result, which partially establishes part 1 of Theorem 7.1.6.

Proposition 7.4.4. *Let $\nu > \frac{n}{r} - 1$, $\mu \in \mathbb{R}$ and $1 \leq p < \infty$ so that*

$$p\nu - \mu > \max(1, p-1)(\tfrac{n}{r} - 1). \tag{7.4.4}$$

If m is sufficiently large (depending only on p and μ) then $P_\nu^{(m)}$ maps L^p_μ boundedly onto $\mathbb{B}^{p,(m)}_\mu$.

Remark 7.4.5. In this proposition it is enough to consider integers m so that $mp + \mu > \max\{1, p-1\}(\frac{n}{r} - 1)$, since in this case $T^+_{\nu,m}$ is bounded in L^p_μ (by Lemma 7.2.5). The result continues to hold as long as $T_{\nu,m}$ is bounded in L^p_μ, for which we give a better range of m and p in Proposition 7.4.22 below. Remark that $\Box^k P_\nu^{(m)} = P_\nu^{(m+k)}$. We could as well speak of the projection P_ν from L^p_μ onto $\mathbb{B}^p_\mu$.

Turning to duality one has the following result.

Proposition 7.4.6. *Let $\mu \in \mathbb{R}$ and $1 < p < \infty$. For any integers $m_1 \geq k_0(p, \mu)$ and $m_2 \geq k_0(p', \mu)$, the dual space $(\mathbb{B}^p_\mu)^*$ identifies with $\mathbb{B}^{p'}_\mu$ under the integral pairing*

$$\langle F, G\rangle_{\mu,m_1,m_2} = \int_{T_\Omega} \Delta^{m_1} \Box^{m_1} F(z) \overline{\Delta^{m_2} \Box^{m_2} G(z)}\, dV_\mu(z), \quad F \in \mathbb{B}^p_\mu, \quad G \in \mathbb{B}^{p'}_\mu. \tag{7.4.5}$$

Moreover, modulo a multiplicative constant, the pairing $\langle \cdot, \cdot\rangle_{\mu,m_1,m_2}$ is independent of m_1 and m_2 satisfying these inequalities.

Proof. The last statement of the theorem follows from the formula of integration by parts in (7.2.8). Thus, we can assume in (7.4.5) that $m_1 = m_2 = m$, for m as large as desired.

If we denote $\Phi_G(F) = \langle F, G\rangle_{\mu,m,m}$, then it is clear that Φ_G defines an element of $(\mathbb{B}^p_\mu)^*$ and that the correspondence $G \in \mathbb{B}^{p'}_\mu \mapsto \Phi_G$ is linear and bounded. To see the injectivity, consider for each $w \in T_\Omega$ the function $F_w = B_{\mu+m}(\cdot - \bar{w})$, which belongs to $\mathbb{B}^p_\mu$ if m is sufficiently large (by Lemma 6.3.2). Then Proposition 7.2.10 gives, for every $G \in \mathbb{B}^{p'}_\mu$, the identity

$$\Phi_G(F_w) \;=\; c \int_{T_\Omega} B_{\mu+2m}(z - \bar{w})\, \overline{\Box^m G(z)}\, \Delta^m(\Im m\, z)\, dV_{\mu+m}(z) \;=\; c\, \overline{\Box^m G(w)},$$

(for large m), from which the injectivity follows easily.

To see the surjectivity, consider $\gamma \in (\mathbb{B}^p_\mu)^*$. Using the isomorphism $\Box^m : \mathbb{B}^p_\mu \to A^p_{\mu+mp}$ (in Proposition 7.4.3) we can define an element $\widetilde{\gamma} \in (A^p_{\mu+mp})^*$ by $\widetilde{\gamma}(H) = \gamma(\Box^{-m} H)$. The functional $\widetilde{\gamma}$ can be extended to $(L^p_{\mu+mp})^*$ by Hahn-Banach, and therefore there exists a function $g \in L^{p'}_\mu$ so that we can write

$$\widetilde{\gamma}(H) = \int H(z)\, \overline{g(z)}\, dV_{\mu+m}(z), \quad H \in A^p_{\mu+mp}.$$

Consequently for every $F \in \mathbb{B}^p_\mu$

$$\gamma(F) = \widetilde{\gamma}(\Box^m F) = \int \Box^m F(z)\,\overline{g(z)}\,dV_{\mu+m}(z).$$

Next, let $G = P^{(m)}_{\mu+m}(g)$ which for large m defines an element of $\mathbb{B}^{p'}_\mu$ (by Proposition 7.4.4). We claim that $\gamma = \Phi_G$. Indeed, when $F \in \mathbb{B}^p_\mu$

$$\begin{aligned}
\langle F, G\rangle_{\mu,m,m} &= \int \Box^m F(z)\,\overline{\Box^m G(z)}\,dV_{\mu+2m}(z)\\
&= c\int \Box^m F(z)\left[\int B_{\mu+2m}(w,z)\overline{g(w)}\,dV_{\mu+m}(w)\right]dV_{\mu+2m}(z)\\
&= c\int\left[\int B_{\mu+2m}(w,z)\,\Box^m F(z)\,\Delta(\Im m\, z)^m\,dV_{\mu+m}(z)\right]\overline{g(w)}\,dV_{\mu+m}(w)\\
\text{(by Proposition 7.2.11)} &= c\int \Box^m F(z)\,\overline{g(w)}\,dV_{\mu+m}(w) \;=\; c\,\gamma(F),
\end{aligned}$$

where Fubini's theorem is justified by the boundedness of the operator $T^+_{\mu+m,m}$ in $L^{p'}_\mu$ when m is sufficiently large. This establishes the claim and completes the proof of the proposition. $\Box$

As a special case we obtain the following, which establishes part 2 of Theorem 7.1.6.

Corollary 7.4.7. *Let $\nu > \frac{n}{r} - 1$ and $1 < p \le 2$. Then, $(A^p_\nu)^*$ identifies with $\mathbb{B}^{p'}_\nu$ under the integral pairing*

$$\langle F, G\rangle_{\nu,m} = \int_{T_\Omega} F(z)\overline{\Delta^m\Box^m G(z)}\,dV_\nu(z), \quad F \in A^p_\nu, \quad G \in \mathbb{B}^{p'}_\nu, \tag{7.4.6}$$

for any integer $m \ge k_0(p',\nu)$.

Proof. Just observe that in this range $k_0(p,\nu) = 0$ and $\mathbb{B}^p_\nu = A^p_\nu$ (see Proposition 7.4.3). $\Box$

Remark 7.4.8. We observe that the duality of Bergman spaces is still open for values of p for which the Hardy inequality is not valid; that is, we do not know any (non trivial) description of the spaces $(A^p_\nu)^*$ for $p \ge p_\nu$.

7.4.3 The Bloch space $\mathbb{B}^\infty(T_\Omega)$

The definition of analytic Besov space and the properties in previous sections extend in an analogous way to the case $p = \infty$, for which $\mathbb{B}^\infty$ is called *Bloch space*. In fact, the Bloch space in T_Ω was already introduced in [7, 8] and shown to be the dual of $A^1(T_\Omega)$. Here we recall these results, together with some new facts about the required number of equivalence classes.

The following inequality is elementary, and can be obtained from the mean value property of holomorphic functions exactly as in [14, Prop. 6.1], so we omit the proof here.

Lemma 7.4.9. *Let $\nu \in \mathbb{R}$. Then*

$$\left\|\Delta(\Im m\,\cdot)^{\nu+1}\Box F\right\|_{L^\infty} \leq C\left\|\Delta(\Im m\,\cdot)^{\nu}F\right\|_{L^\infty}, \quad \forall\, F \in \mathcal{H}(T_\Omega). \tag{7.4.7}$$

For every integer m we define a Bloch type space

$$\mathbb{B}^{\infty,(m)} := \left\{F \in \mathcal{H}_m(T_\Omega) \;:\; \Delta^m\Box^m F \in L^\infty\right\},$$

endowed with the norm $\|F\|_{\mathbb{B}^{\infty,(m)}} = \|\Delta^m\Box^m F\|_\infty$. We simply write $\mathbb{B}^\infty(T_\Omega)$ for the space $\mathbb{B}^{\infty,(m)}$ with $m = \lceil \frac{n}{r} - 1\rceil$, the smallest integer greater than $\frac{n}{r} - 1$. We have the following property:

Proposition 7.4.10. *For all integers $m \geq k > \frac{n}{r} - 1$, the natural inclusion of $\mathbb{B}^{\infty,(k)}$ into $\mathbb{B}^{\infty,(m)}$ is an isomorphism of Banach spaces.*

Proof. We may assume $m = k + 1$. By Lemma 7.4.9

$$\left\|\Delta^{k+1}\Box^{k+1}f\right\|_{L^\infty} \leq C\left\|\Delta^k\Box^k f\right\|_{L^\infty}, \quad f \in \mathbb{B}^{\infty,(k)}.$$

We want to prove the converse inequality, which is the analogue of Hardy's inequality for $p = \infty$, that is,

$$\|\Delta^k\Box^k f\|_\infty \leq C\|\Delta^{k+1}\Box^{k+1}f\|_\infty \tag{7.4.8}$$

for all $k > \frac{n}{r} - 1$ and all $f \in \mathcal{H}(T_\Omega)$ for which the left hand side is finite. Choosing $\nu > \frac{n}{r} - 1$, we may use Proposition 7.2.10 to write

$$\Box^k f = c\int_{T_\Omega} B_{\nu+k}(\cdot - \bar{w})\Box^{k+1}f(w)\,\Delta^{k+1}(\Im m\, w)\,dV_\nu(w). \tag{7.4.9}$$

The inequality (7.4.8) follows from the fact that $\int_{T_\Omega}|B_{\nu+k}(z-\bar{w})|dV_\nu(w) \leq C\Delta^{-k}(\Im m\, z)$ by Lemma 6.3.2.

This implies the injectivity of the mapping. Let us finally prove that the mapping is onto. Let $f \in \mathcal{H}(T_\Omega)$ be such that $\Delta^{k+1}\Box^{k+1}f$ is bounded. Then the right hand side of (7.4.9) defines a holomorphic function, which may be written as $\Box^k g$. We prove as before that $\Delta^k\Box^k g$ is bounded. Moreover, $\Box^{k+1}g = \Box^{k+1}f$, which proves the surjectivity of the mapping. $\square$

Remark 7.4.11. Observe that when $k \le \frac{n}{r} - 1$, the injectivity of $\mathbb{B}^{\infty,(k)} \to \mathbb{B}^{\infty,(m)}$ fails. Indeed, the function $F(z) = \Delta^{k+1-\frac{n}{r}}(z + i\mathbf{e})$ belongs to $\mathbb{B}^{\infty,(k)}$ and is typically not null in $\mathcal{H}_k$. However, F is zero in $\mathcal{H}_m$ for all $m > k$ since by (7.2.3) and (7.2.4), we have $\Box^{k+1}F(z) = c\Box\Delta^{1-\frac{n}{r}}(z + i\mathbf{e}) = 0$.

Remark 7.4.12. We do not know whether for some $k \le \frac{n}{r}-1$ the correspondence $\mathbb{B}^{\infty,(k)} \to \mathbb{B}^{\infty,(m)}$ may be surjective. This question can also be formulated as follows: *Is it possible that every element f of $\mathbb{B}^\infty$ possesses a representative g such that $\Delta^k\Box^k g$ is bounded, with $k \le \frac{n}{r} - 1$?*. We shall answer partially this question in Section 7.4.6. It seems that this problem has never been considered before in the literature.

We now turn to the boundedness of Bergman operators in L^∞. As we did in Subsection 7.4.2, when $\nu > \frac{n}{r} - 1$ we may extend the definition of the Bergman projection P_ν to L^∞ functions by letting $P_\nu^{(m)} f$ be the equivalence class (in $\mathcal{H}_m$) of all holomorphic solutions of

$$\Box^m F \;=\; c_{\nu,m} \int_{T_\Omega} B_{\nu+m}(\cdot - \bar{w}) f(w)\, dV_\nu(w).$$

To do this, it suffices to consider $m > \frac{n}{r} - 1$, since by Lemma 7.2.6 the above integral is always absolutely convergent and moreover

$$\|P_\nu^{(m)} f\|_{\mathbb{B}^{\infty,(m)}} \;=\; \|T_{\nu,m} f\|_\infty \lesssim \|f\|_\infty.$$

Thus $P_\nu^{(m)}$ maps $L^\infty \to \mathbb{B}^\infty$ boundedly. The mapping is surjective, as every $F \in \mathbb{B}^\infty$ satisfies (by Proposition 7.2.10)

$$\Box^m F(z) \;=\; \int_{T_\Omega} B_{\nu+m}(z, w)\, \Box^m F(w)\, \Delta(\Im m\, w)^m\, dV_\nu(w)$$

and therefore $F = cP_\nu^{(m)}(f)$ with $f = \Delta^m\Box^m F \in L^\infty$. Hence we have established the following result.

Proposition 7.4.13. *When $\nu, m > \frac{n}{r} - 1$, the Bergman projection $P_\nu^{(m)}$ maps $L^\infty(T_\Omega)$ continuously onto $\mathbb{B}^\infty$.*

Concerning duality, we recall the identification of the Bloch space with the dual of the Bergman space A^1_ν.

Theorem 7.4.14 (Békollé, [8]). *Let $\nu, m > \frac{n}{r} - 1$. Then the dual space $(A^1_\nu)^*$ identifies with the Bloch space $\mathbb{B}^\infty$ under the integral pairing*

$$\langle F, G\rangle_{\nu,m} = \int_{T_\Omega} F(z)\overline{\Delta(\Im m\, z)^m \Box^m G(z)} dV_\nu(z), \quad F \in A^1_\nu,\;\; G \in \mathbb{B}^\infty. \tag{7.4.10}$$

Moreover, the pairing $\langle \cdot, \cdot \rangle_{\nu,m}$ *is independent of* $m > \frac{n}{r} - 1$.

The proof is entirely analogous to the one presented in Proposition 7.4.6, so we omit it. Let now $\mu \in \mathbb{R}$. Since $\Box^m : \mathbb{B}^1_\mu \to A^1_{\mu+m}$ is an isomorphism when $\mu + m > \frac{n}{r} - 1$ (by Proposition 7.4.3), we obtain as a corollary the following duality statement.

Corollary 7.4.15. *Let* $\mu \in \mathbb{R}$ *and let* m_1, m_2 *be two integers such that* $\mu + m_1 > \frac{n}{r} - 1$ *and* $m_2 > \frac{n}{r} - 1$. *Then* $(\mathbb{B}^1_\mu)^*$ *identifies with the Bloch space* $\mathbb{B}^\infty$ *under the integral pairing*

$$\langle F, G \rangle_{\mu,m_1,m_2} = \int_{T_\Omega} L_{m_1} F(z) \overline{L_{m_2} G(z)} \Delta^{\mu - \frac{n}{r}}(\Im z) dz, \quad F \in \mathbb{B}^1_\mu, \;\; G \in \mathbb{B}^\infty,$$

where $L_m H(z) = \Delta^m(Im z) \Box^m_z H(z)$. *Again, the pairing* $\langle \cdot, \cdot \rangle_{\mu,m_1,m_2}$ *is independent of* m_1, m_2 *(modulo a multiplicative constant).*

7.4.4 A real analysis characterization of $\mathbb{B}^p_\mu$

We briefly recall the real variable theory of Besov spaces adapted to the cone that was developed in [13].

Following [13, Section 3], we consider a *lattice* $\{\xi_j\}$ in Ω and a sequence $\{\psi_j\}$ of Schwartz functions in $\mathbb{R}^n$ such that $\widehat{\psi}_j$ is supported in an invariant ball centered at ξ_j and $\sum_j \widehat{\psi}_j = \chi_\Omega$. In particular, the sets Supp $\widehat{\psi}_j$ have the finite intersection property and the norms $\|\psi_j\|_{L^1(\mathbb{R}^n)}$ are uniformly bounded. Below we denote by $\mathcal{S}'_{\partial\Omega}$ the space of tempered distributions with Fourier transform supported in $\partial\Omega$. Observe that $\Box u = 0$ (in $\mathcal{S}'$) implies Supp $\hat{u} \subset \partial\Omega \cup (-\partial\Omega)$.

Definition 7.4.16. Given $\nu \in \mathbb{R}$ and $1 \le p < \infty$, we define

$$B^p_\nu := \big\{ f \in \mathcal{S}'(\mathbb{R}^n) \;:\; \text{Supp } \widehat{f} \subset \overline{\Omega} \quad \text{and} \quad ||f||_{B^p_\nu} < \infty \big\} / \mathcal{S}'_{\partial\Omega},$$

where the seminorm is given by

$$||f||_{B^p_\nu} := \big(\sum_j \Delta^{-\nu}(\xi_j) \, ||f * \psi_j||^p_p \big)^{\frac{1}{p}}.$$

It can be shown that B^p_ν is a Banach space and the definition is independent on the choice of $\{\xi_j, \psi_j\}$ (see [13]). In the 1-dimensional setting B^p_ν coincides with the classical homogeneous Besov space $\dot{B}^{-\nu/p}_{p,p}(\mathbb{R})$ (of distributions with spectrum in $[0, \infty)$, modulo polynomials).

In certain cases one can avoid equivalence classes in Definition 7.4.16, and this will turn into a representation of $\mathbb{B}^p_\nu$ as a holomorphic function space. We denote by $\mathcal{L}g(z) =$

$(g, e^{i(z|\cdot)})$, $z \in T_\Omega$, the *Fourier-Laplace transform* of a distribution g compactly supported in Ω (which defines an analytic function in T_Ω). For convenience, we write Υ for the set of indices (p, ν) such that

$$\nu > -\tfrac{n}{r} \quad \text{and} \quad 1 \le p < \tilde{p}_\nu, \quad \text{or} \quad \nu = -\tfrac{n}{r} \quad \text{and} \quad p = \tilde{p}_\nu = 1. \tag{7.4.11}$$

Then, in [13, Lemmas 3.38 and 3.43] the following result is shown.

Lemma 7.4.17. *Let $(p, \nu) \in \Upsilon$. Then if $f \in B^p_\nu$*

(i) the series $\sum_j f * \psi_j$ converges in $\mathcal{S}'(\mathbb{R}^n)$ to a distribution $f^\sharp$;

(ii) the series $\sum_j \mathcal{L}(\widehat{f}\widehat{\psi}_j)(z)$ converges uniformly on compact sets to a holomorphic function in T_Ω, denoted by $\mathcal{E}(f)(z)$, which satisfies

$$\Delta(\Im m\, z)^{(\nu + \frac{n}{r})/p} \, |\mathcal{E}(f)(z)| \le C \, \|f\|_{B^p_\nu}, \quad z \in T_\Omega.$$

In addition, the mappings

$$f \in B^p_\nu \longrightarrow f^\sharp \in \mathcal{S}'(\mathbb{R}^n) \quad \text{and} \quad f \in B^p_\nu \longrightarrow \mathcal{E}(f) \in \mathcal{H}(T_\Omega)$$

are continuous and injective, and for every $f \in B^p_\nu$ we have

$$\lim_{\substack{y \to 0 \\ y \in \Omega}} \mathcal{E}(f)(\cdot + iy) = f^\sharp \quad \text{in } \mathcal{S}'(\mathbb{R}^n) \text{ and in } \| \cdot \|_{B^p_\nu}.$$

Remark 7.4.18. The results in [13] are stated only for $\nu > 0$, but remain valid as long as $(p, \nu) \in \Upsilon$.

From this lemma we can define an isometric copy of B^p_ν (and hence of $\mathbb{B}^p_\nu$) as a holomorphic function space in $\mathcal{H}(T_\Omega)$:

Definition 7.4.19. For $(p, \nu) \in \Upsilon$ we define the holomorphic function space

$$\mathcal{B}^p_\nu := \{F = \mathcal{E} f : f \in B^p_\nu\},$$

endowed with the norm $||F||_{\mathcal{B}^p_\nu} = ||f||_{B^p_\nu}$.

The following properties hold

(a) $\mathcal{B}^p_\nu = A^p_\nu$ when Hardy's inequality holds for (p, ν), and in particular when $\nu > \frac{n}{r} - 1$ and $1 \le p < \bar{p}_\nu$ (see [13, p. 351]).

(b) $A^p_\nu \hookrightarrow \mathcal{B}^p_\nu$ when $\nu > \frac{n}{r} - 1$ and $1 \le p < \tilde{p}_\nu$. The inclusion is strict in the 3-dimensional light-cone when $\nu < 1$ and $p_\nu \le p < \tilde{p}_\nu$.

(c) $\mathcal{B}^2_0 = H^2(T_\Omega)$ (Hardy space). Moreover, $\Big\{\mathcal{B}^2_\nu = \mathcal{L}\Big(L^2(\Omega;\Delta^{-\nu}(\xi)\,d\xi)\Big)\Big\}_{\nu>-1}$ is the family of spaces introduced by Vergne and Rossi in the study of irreducible representations of the group of conformal transformations of T_Ω (see [107] or [40, Ch. XIII]).

(d) If $(p,\nu)\in\Upsilon$ then $\Box : \mathcal{B}^p_\nu \to \mathcal{B}^p_{\nu+p}$ is an isomorphism of Banach spaces. This is inherited from the corresponding property in the scale B^p_ν (see [13, Th. 1.4]).

(e) If $(p,\nu)\in\Upsilon$ then $\mathbb{B}^p_\nu$ can be identified with $\mathcal{B}^p_\nu$, in the sense that every $F\in\mathbb{B}^p_\nu$ has a (unique) representative $\widetilde{F}$ in $\mathcal{B}^p_\nu$, and moreover $\|F\|_{\mathbb{B}^p_\nu}\approx\|\widetilde{F}\|_{\mathcal{B}^p_\nu}$. To show this, let $m=k_0(p,\nu)$ so that $\Box^m F\in A^p_{\nu+mp}=\mathcal{B}^p_{\nu+mp}$ (by (a)). Then use (d) to define the unique $\widetilde{F}\in\mathcal{B}^p_\nu$ such that $\Box^m\widetilde{F}=\Box^m F$.

The assertion in (e) above gives a representation of $\mathbb{B}^p_\nu$ as a holomorphic function space with no equivalence classes involved. For example, when $\nu=-n/r$, the space $\mathbb{B}^1$ can be represented by the holomorphic function space $\mathcal{B}^1_{-n/r}$, even in the one-dimensional setting.

Using the box operator, this procedure can be easily extended to all indices (p,ν) (not necessarily in Υ), to represent $\mathbb{B}^p_\nu$ with less equivalence classes than $k_0(p,\nu)$. Namely, given $\nu\in\mathbb{R}$ and $1\le p<\infty$, define

$$k_* = k_*(p,\nu) \;=\; \min\big\{k\in\mathbb{N} \;:\; (p,\nu+kp)\in\Upsilon\big\}. \tag{7.4.12}$$

Observe that $k_*(p,\nu)\le k_0(p,\nu)$, and the inequality is often strict. In fact,

$$k_*(p,\nu)=\min\big\{k \;:\; k+\tfrac{\nu}{p}>(\tfrac{n}{r}-1)(1-\tfrac{2}{p})-\tfrac{1}{p}\big\}$$

(and $k_*(1,\nu)=\min\{k \;:\; k+\nu\ge-\frac{n}{r}\}$). Then we have the following result.

Proposition 7.4.20. *Let $\nu\in\mathbb{R}$, $1\le p<\infty$ and $k_*(p,\nu)$ defined as in* (7.4.12)*. Then every $F\in\mathbb{B}^p_\nu$ has a unique representative $\widetilde{F}$, modulo $\mathcal{N}_{k_*}$, such that $\Box^{k_*}\widetilde{F}\in\mathcal{B}^p_{\nu+k_*p}$, and moreover $\|F\|_{\mathbb{B}^p_\nu}\approx\|\Box^{k_*}\widetilde{F}\|_{\mathcal{B}^p_{\nu+k_*p}}$. In particular, $\mathbb{B}^p_\nu$ identifies with the space*

$$\{G\in\mathcal{H}_{k_*} \;:\; \Box^{k_*}G\in\mathcal{B}^p_{\nu+k_*p}\}. \tag{7.4.13}$$

Proof: Combine the fact that $\mathbb{B}^p_{\nu+k_*p}$ identifies with $\mathcal{B}^p_{\nu+k_*p}$ (by property (e) above), with the trivial isomorphism $\Box^{k_*}:\mathbb{B}^p_\nu\to\mathbb{B}^p_{\nu+k_*p}$.

$\Box$

We turn now to the identification between the spaces $\mathbb{B}^p_\nu$ and B^p_ν via boundary values, as asserted in the introduction. When $(p,\nu)\in\Upsilon$ the result is immediate from (e) above.

Corollary 7.4.21. *Let $(p,\nu)\in\Upsilon$. Then*

(i) if $F\in\mathbb{B}^p_\nu$, there exists $\lim_{\substack{y\to 0\\ y\in\Omega}}\widetilde{F}(\cdot+iy)=f$ in B^p_ν (and $\mathcal{S}'$), for some representative $\widetilde{F}$ of F.

(ii) if $f\in B^p_\nu$, there exists a unique $F\in\mathbb{B}^p_\nu$ such that $\lim_{\substack{y\to 0\\ y\in\Omega}}F(\cdot+iy)=f$ in B^p_ν.

In either case

$$\tfrac{1}{c}\,\|f\|_{B^p_\nu}\;\leq\;\|F\|_{\mathbb{B}^p_\nu}\;\leq\;c\,\|f\|_{B^p_\nu}.$$

The inverse mapping in (ii) is defined by the operator $f\mapsto F=\mathcal{E}(f)$. For general parameters p and ν, $\mathcal{E}f$ is no longer defined when $f\in B^p_\nu$, but $\mathcal{E}(\Box^{k_*}f)$ is well-defined and belongs to $\mathcal{B}^p_{\nu+k_*p}$. Thus, using Proposition 7.4.20, we may consider a new operator $\mathtt{E}$ from B^p_ν into $\mathbb{B}^p_\nu$ by

$$\Box^{k_*}\mathtt{E}f:=\mathcal{E}(\Box^{k_*}f).$$

It is easily seen that $\mathtt{E}:B^p_\nu\to\mathbb{B}^p_\nu$ is an isomorphism, which commutes with the Box operator

$$\Box^\ell_z\circ\mathtt{E}=\mathtt{E}\circ\Box^\ell_x,\qquad\forall\,\ell\in\mathbb{N}.$$

Moreover, duality can be expressed through this isomorphism. Recall first that (see [13])

$$(B^p_\nu)^*=B^{p'}_\nu$$

whenever the definition of the duality pairing is given by

$$[f,g]_\nu:=\sum_j\langle f,\Box^{-\nu}g*\psi_j\rangle,\qquad f\in B^p_\nu,\quad g\in B^{p'}_\nu. \tag{7.4.14}$$

On the right hand side the brackets stand for the action of the distribution f on the conjugate of the given test function, while $\Box^{-\nu}$ is defined on the Fourier side by the multiplication by $\Delta(\xi)^{-\nu}$. Then, the duality result in Proposition 7.4.6 can also be obtained from the above discussion, since when $F=\mathtt{E}f\in\mathbb{B}^p_\mu$, $G=\mathtt{E}g\in\mathbb{B}^{p'}_\mu$ and m is large we have

$$\langle F,G\rangle_{\mu,m,m}=c_{m,\mu}\,[f,g]_\mu.$$

Finally, using real variable techniques we are able to improve on the results in Proposition 7.4.4 concerning the range of p and number m for which there is boundedness of $P^{(m)}_\nu$ from L^p_μ into $\mathbb{B}^p_\mu$. Below we consider P_ν as a densely defined operator in $L^p_\mu\cap L^2_\nu$.

Proposition 7.4.22. *Let $\nu > \frac{n}{r} - 1$, $\mu \in \mathbb{R}$ and $1 \le p < \infty$ so that*

$$p\nu - \mu > \max\{p-1, 2-p\}\,(\tfrac{n}{r} - 1). \tag{7.4.15}$$

If $k_* = k_*(p,\mu)$ is as in (7.4.12), then $\Box^{k_*} \circ P_\nu$ extends as a bounded surjective mapping from L^p_μ onto $\mathcal{B}^p_{\mu+k_*p}$.

Remark 7.4.23. As a special case we obtain that, in the range in (7.4.15), $P_\nu^{(k_0)}$ maps L^p_μ continuously onto $\mathbb{B}^p_\mu$, which in particular establishes part 1 of Theorem 7.1.6. Equivalently, the operator $T_{\nu,m}$ given by (7.2.6) is bounded in L^p_μ for all $m \ge k_0(p,\mu)$; see the discussion preceeding Proposition 7.4.4.

When $\mu = -n/r$ the condition (7.4.15) produces no restriction in p, and we obtain the following.

Corollary 7.4.24. *For all $\nu > \frac{n}{r} - 1$ and $1 \le p < \infty$, the operator $P_\nu^{(k_0)}$ maps $L^p(T_\Omega, d\lambda)$ onto $\mathbb{B}^p$. Moreover, P_ν extends boundedly from $L^1(d\lambda)$ onto $\mathcal{B}^1$.*

Proof of Proposition 7.4.22: The continuity follows from a similar reasoning as in [13, Prop. 4.28], where the case $\mu = \nu$ was proved. For completeness, we sketch here the modifications of the general case. Given $f \in L^p_\mu \cap L^2_\nu$, since $P_\nu f \in A^2_\nu$ we can write it, by the Paley-Wiener theorem, as $P_\nu f = \mathcal{L}g$, for some $g \in L^2(\Omega, \Delta^{-\nu}(\xi)d\xi)$. We must show that $\Box^{k_*} P_\nu f = \mathcal{L}(\Delta^{k_*} g)$ belongs to $\mathcal{B}^p_{\mu+k_*p}$, or equivalently that the inverse Fourier transform of the distribution $\Delta^{k_*} g$ belongs to the real space $B^p_{\mu+k_*p}$. Arguing by duality as in (7.4.14), this is equivalent to showing that for all smooth φ with compact spectrum in Ω

$$\left| \langle \Delta^{k_*} g, \Delta^{-\mu-k_*p} \widehat{\varphi} \rangle \right| \;\le\; C \, \|f\|_{L^p_\mu} \, \|\varphi\|_{B^{p'}_{\mu+k_*p}}.$$

By the Paley-Wiener theorem for Bergman spaces (see eg [40, p.260])

$$\begin{aligned}
LHS \quad &= \quad \int_\Omega g(\xi)\, \Delta^{-\mu-k_*(p-1)}(\xi)\, \overline{\widehat{\varphi}(\xi)}\, \tfrac{\Delta^\nu(\xi)}{\Delta^\nu(\xi)}\, d\xi \\
&= \quad \iint_{T_\Omega} P_\nu f(w)\, \overline{\mathcal{E}(\Box^{\nu-\mu-k_*(p-1)}\varphi)}(w)\, dV_\nu(w) \\
\text{(since } P_\nu^* = P_\nu) \quad &= \quad \langle f, \mathcal{E}(\Box^{\nu-\mu-k_*(p-1)}\varphi)\rangle_{dV_\nu} \;\le\; \|f\|_{L^p_\mu} \, \|\Delta^{\nu-\mu}\mathcal{E}(\Box^{\nu-\mu-k_*(p-1)}\varphi)\|_{L^{p'}_\mu}.
\end{aligned}$$

If $p > 1$ the last norm equals

$$\|\mathcal{E}(\Box^{\nu-\mu-k_*(p-1)}\varphi)\|_{L^{p'}_{(\nu-\mu)p'+\mu}}.$$

Under the conditions (7.4.15) we have $A^{p'}_{(\nu-\mu)p'+\mu} = \mathcal{B}^{p'}_{(\nu-\mu)p'+\mu}$, since Hardy's inequality holds for the corresponding indices. Thus,

$$\|\mathcal{E}(\Box^{\nu-\mu-k_*(p-1)}\varphi)\|_{A^{p'}_{(\nu-\mu)p'+\mu}} \approx \|\Box^{\nu-\mu-k_*(p-1)}\varphi\|_{B^{p'}_{(\nu-\mu)p'+\mu}} \lesssim \|\varphi\|_{B^{p'}_{\mu+k_*p}},$$

as we wished to prove. When $p = 1$ one must use instead

$$\|\Delta^{\nu-\mu}\mathcal{E}(\Box^{\nu-\mu}\varphi)\|_{L^\infty} \lesssim \|\Box^{\nu-\mu}\varphi\|_{B^\infty_{\nu-\mu}} \approx \|\varphi\|_{B^\infty_0}$$

(see Lemma 7.4.26 below), and conclude again by duality. The surjectivity of $\Box^{k_*} \circ P_\nu$ follows from the surjectivity of the operator $P^{(m)}_\nu : L^p_\mu \to \mathbb{B}^{p,(m)}_\mu$ for large m in Proposition 7.4.4, since the spaces $\mathcal{B}^p_{\mu+k_*p}$ and $\mathbb{B}^{p,(m)}_\mu$ are related by isomorphisms.

□

7.4.5 A real variable characterization of $\mathbb{B}^\infty$

For completeness, we give here the real variable characterization of the Bloch space $\mathbb{B}^\infty$, starting with the definition of the distribution spaces B^∞_ν introduced in [13].

Definition 7.4.25. For $\nu \in \mathbb{R}$ we let

$$||f||_{B^\infty_\nu} = \sup_j \Delta(\xi_j)^{-\nu}||f * \psi_j||_\infty, \quad f \in S'(\mathbb{R}^n),$$

and define the space B^∞_ν by

$$B^\infty_\nu := \{f \in \mathcal{S}'(\mathbb{R}^n) \ : \ \operatorname{Supp} \widehat{f} \subset \overline{\Omega} \quad \text{and} \quad ||f||_{B^\infty_\nu} < \infty\}/\mathcal{S}'_{\partial\Omega}.$$

The following result is the analogue of Lemma 7.4.17 for $p = \infty$. The result was not stated in [13], so we sketch the proof for completeness.

Lemma 7.4.26. *Let $\nu > \frac{n}{r} - 1$ and $f \in B^\infty_\nu$. Then*

(i) $\sum_j f * \psi_j$ converges in $\mathcal{S}'(\mathbb{R}^n)$ to a distribution $f^\sharp$;

(ii) $\sum_j \mathcal{L}(\widehat{f}\widehat{\psi}_j)(z)$ converges uniformly on compact sets of T_Ω to a holomorphic function $\mathcal{E}(f)(z)$, which satisfies

$$\Delta(\Im m\, z)^\nu\, |\mathcal{E}(f)(z)| \leq C\, \|f\|_{B^\infty_\nu}, \quad z \in T_\Omega.$$

Proof. By duality, (i) is equivalent to $\mathcal{S}(\mathbb{R}^n) \hookrightarrow B^1_{-\nu}$, which in view of [13, Prop 3.16] happens if and only if $\nu > \frac{n}{r} - 1$. Concerning (ii) and reasoning as in the proof of [13, Prop 3.43], it suffices to see that $\mathcal{F}^{-1}(e^{-(\mathbf{e}|\cdot)}\chi_\Omega)$ belongs to the space $B^1_{-\nu}$. Using the isomorphism $\Box^{2\nu}$ and the identity $\mathcal{B}^1_\nu = A^1_\nu$ this is equivalent to $\mathcal{L}(\Delta^{2\nu}e^{-(\mathbf{e}|\cdot)}\chi_\Omega)(z) = c\Delta(z + i\mathbf{e})^{-2\nu-\frac{n}{r}} \in A^1_\nu$, which by Lemma 6.3.2 happens if and only if $\nu > \frac{n}{r} - 1$. □

For simplicity we denote $B^\infty = B_0^\infty$, which can be identified with the Bloch space $\mathbb{B}^\infty$ as follows.

Proposition 7.4.27. *For all $k > \frac{n}{r} - 1$, the correspondence*

$$f \in B^\infty \longmapsto \Box_z^{-k}\big[\mathcal{E}(\Box^k f)\big] \in \mathbb{B}^\infty$$

is an isomorphism of Banach spaces.

Proof. Since $\Box^k f \in B_k^\infty$, by the previous lemma the function $G := \mathcal{E}(\Box^k f)$ is holomorphic in T_Ω and $\Delta^k(\Im m\, z)G(z)$ is bounded. Thus the equivalence class of all F such that $\Box_z^k F = G$ belongs to $\mathbb{B}^\infty$, and the correspondence $f \mapsto F + \mathcal{N}_k$ defines a bounded operator from B^∞ to $\mathbb{B}^\infty$.

On the other hand, whenever $\nu > \frac{n}{r} - 1$ and $H := \mathcal{E}(h)$ is in A_ν^1, so that h belongs to B_ν^1, one has

$$\int_{T_\Omega} H(z)\overline{\Box^k F(z)}\Delta^k(\Im z)dV_\nu(z) = [h, f]_\nu.$$

Using the duality identities $\mathbb{B}^\infty = (A_\nu^1)^*$ (with the above pairing) and $B^\infty = (B_\nu^1)^*$ (with the pairing $[\cdot,\cdot]_\nu$), it follows that the mapping $f \mapsto F$ is an isomorphism, like the mapping $h \mapsto H$. $\Box$

7.4.6 Minimum number of equivalence classes: partial results

Related with the question of the smallest number of derivatives in the definition of $\mathbb{B}_\nu^p$, one can also consider a weaker property than Hardy's inequality; namely

Question: *Given $1 \le p \le \infty$ and $\nu \in \mathbb{R}$, find the smallest $\ell = \ell(p,\nu) \in \mathbb{N}$ so that, for all $m \ge 1$,*

$$\inf_{H \in \mathcal{H}(T_\Omega)\ :\ \Box^{\ell+m} H = 0} \big\|\Delta^\ell \Box^\ell (F + H)\big\|_{L_\nu^p} \lesssim \big\|\Delta^{\ell+m}\Box^{\ell+m} F\big\|_{L_\nu^p}, \tag{7.4.16}$$

for all holomorphic F for which the right hand side is finite.

We look first at $p = \infty$ and its equivalent formulation raised in Remark 7.4.12, namely the surjectivity of the mapping $\mathbb{B}^{\infty,(k)} \to \mathbb{B}^\infty$ for $k \le \frac{n}{r} - 1$. We prove that it cannot happen at least when $k \le (\frac{n}{r} - 1)/2$.

Proposition 7.4.28. *Let k be a non negative integer. If, for every $F \in \mathbb{B}^\infty$, there exists $\widetilde{F}$ such that $\Delta^k \Box^k \widetilde{F}$ is bounded and $\Box^m \widetilde{F} = \Box^m F$ for some $m > \frac{n}{r} - 1$, then necessarily $k > \frac{1}{2}(\frac{n}{r} - 1)$.*

Proof. Let $m > \frac{n}{r} - 1$. By the open mapping theorem, if this property is valid, the natural mapping of $\mathbb{B}^{\infty,(k)}$ into $\mathbb{B}^{\infty,(m)}$, which is surjective, defines an isomorphism from the quotient space $\mathbb{B}^{\infty,(k)}/\mathcal{N}_m$ onto $\mathbb{B}^{\infty,(m)}$. So there is some constant C such that, for each $F \in \mathbb{B}^{\infty,(m)}$, there exists some G with $\Box^m G = 0$ and

$$\|F + G\|_{\mathbb{B}^{\infty,(k)}} \leq C\|F\|_{\mathbb{B}^{\infty,(m)}}.$$

In particular,

$$|\Box^k F(x + i\mathbf{e}) + \Box^k G(x + i\mathbf{e})| \leq C\|F\|_{\mathbb{B}^{\infty,(m)}}.$$

Consider now $F = \mathcal{E}f$ with $\widehat{f} \in C_c^\infty(\Omega)$, so that $\|F\|_{\mathbb{B}^{\infty,(m)}} \leq C\|f\|_{B^\infty}$. Since $\Box^k F(x+i\mathbf{e})$ is bounded, the same is valid for $\Box^k G(x + i\mathbf{e})$. So we can speak of the Fourier transform of $\Box^k G(x+i\mathbf{e})$, whose support is in the boundary of Ω. Let φ be a smooth function whose Fourier transform is compactly supported in Ω, and consider its scalar product, in the x variable, with the function $\Box^k F(x+i\mathbf{e})+\Box^k G(x+i\mathbf{e})$. By the support condition on $\hat{\varphi}$ we must have $\langle \Box^k G(x + i\mathbf{e}), \varphi\rangle = 0$. So, the following inequality, valid for all such F, holds

$$\left| \int_{\mathbb{R}^n} \Box^k F(x + i\mathbf{e})\overline{\varphi(x)}dx \right| \leq C\|f\|_{B^\infty} \times \|\varphi\|_1.$$

The last inequality can as well be written as

$$\left| \int_{\mathbb{R}^n} f(x)\overline{T\varphi(x)}dx \right| \leq C\|f\|_{B^\infty} \times \|\varphi\|_1,$$

where $\widehat{(T\varphi)}(\xi) = \Delta(\xi)^k e^{-(\mathbf{e}|\xi)}\hat{\varphi}(\xi)$. In view of the duality $(B_0^1)^* = B^\infty$, it is easily seen that this implies the inequality

$$\|T\varphi\|_{B_0^1} \leq C\|\varphi\|_1. \tag{7.4.17}$$

We want to find a contradiction by choosing specific functions φ. Assume that $\varphi := \varphi_t$ may be written as

$$\varphi_t(x) = \sum_{j\in J} r_j(t)a_j e^{i(x|\xi_j)}\eta(x),$$

where J is a finite set of indices, and η is a smooth function whose Fourier transform is supported in a small ball centered at 0, in such a way that the functions ψ_j can be assumed to be equal to 1 on the support of $\hat{\eta}(\cdot - \xi_j)$, for all $j \in J$. Here $r_k(t)$ stands for the Rademacher function and the parameter t varies in $(0, 1)$. Integrating in t and using Khintchine's Inequality, we have

$$\int_0^1 \|T\varphi_t\|_{B^1} dt \leq C \int_0^1 \|\varphi_t\|_1 dt \leq C' \left(\sum_{j\in J} |a_j|^2 \right)^{1/2} \|\eta\|_1. \tag{7.4.18}$$

Let us find a minorant for the left hand side of (7.4.18). For every choice of t, we have

$$\|T\varphi_t\|_{B^1} = \sum_{j\in J} |a_j| \, \big\|T(e^{i(\cdot|\xi_j)}\eta)\big\|_1.$$

Let us take for granted the existence of some uniform constants $c_1, c_2 > 0$ such that

$$\big\|T(e^{i(\cdot|\xi_j)}\eta)\big\|_1 = \Big\|\mathcal{F}^{-1}[\Delta^k e^{-(\mathbf{e}|\cdot)}\hat{\eta}(\cdot - \xi_j)]\Big\|_1 \geq \frac{1}{c_1}\Delta(\xi_j)^k \, e^{-c_2(\mathbf{e}|\xi_j)} \, \|\eta\|_1. \tag{7.4.19}$$

Then, (7.4.18) leads to the existence of some (different) constant C such that

$$\sum_{j\in J} |a_j| \Delta(\xi_j)^k e^{-c_2(\mathbf{e}|\xi_j)} \leq C \left(\sum_{j\in J} |a_j|^2\right)^{1/2}.$$

We choose $a_j = \Delta(\xi_j)^k e^{-c_2(\mathbf{e}|\xi_j)}$ and find that

$$\sum_{j\in J} \Delta(\xi_j)^{2k} e^{-2c_2(\mathbf{e}|\xi_j)} \leq C^2$$

uniformly when J varies among finite sets of indices. This allows to have the same estimate for the sum over all indices j, that is

$$\sum_{j} \Delta(\xi_j)^{2k} e^{-2c_2(\mathbf{e}|\xi_j)} < \infty.$$

By [13, Prop. 2.13] this sum behaves as the integral

$$\int_\Omega \Delta(\xi)^{2k} e^{-(\mathbf{e}|\xi)} \frac{d\xi}{\Delta(\xi)^{n/r}},$$

which is finite for $2k > \frac{n}{r} - 1$.

It remains to prove our claim (7.4.19), which we do by using group action as in [13, (3.47)]. Write $\xi_j = g_j \mathbf{e}$ with $g_j = g_j^* \in G$, and let $\chi_j(\xi) = \chi(g_j^{-1}\xi)$ for some $\chi \in C_c^\infty(\Omega)$ with the property that $\chi_j \equiv 1$ in Supp $\hat{\eta}(\cdot - \xi_j)$, $\forall\, j \in J$ (which we can do by our choice of η). Consider the function γ_j whose Fourier transform is defined by

$$\widehat{\gamma_j}(\xi) := e^{(\mathbf{e}|\xi)} \Delta(\xi)^{-k} \chi_j(\xi),$$

so that we can write

$$e^{i(\cdot|\xi_j)}\eta = \gamma_j * T\big(e^{i(\cdot|\xi_j)}\eta\big), \quad \forall\, j \in J.$$

Thus, it suffices to show that

$$\|\gamma_j\|_1 \leq c_1 \Delta(\xi_j)^{-k} e^{c_2(\mathbf{e}|\xi_j)}. \tag{7.4.20}$$

Now, a change of variables gives

$$\|\gamma_j\|_1 \;=\; \big\|\,\mathcal{F}^{-1}\big[\,e^{(\mathbf{e}|g_j\xi)}\Delta(g_j\xi)^{-k}\chi(\xi)\,\big]\,\big\|_1 \;=\; \Delta(\xi_j)^{-k}\,\big\|\,\mathcal{F}^{-1}\big[\,e^{(\xi_j|\cdot)}\Delta^{-k}\chi\,\big]\,\big\|_1,$$

where in the last equality we have used (7.2.1) and $g_j^* = g_j$. The L^1-norm on the right hand side can be controlled by a Schwartz norm of $e^{(\xi_j|\cdot)}\Delta^{-k}\chi$, which leads to (7.4.20) using the fact that $e^{(\xi_j|\xi)} \le e^{c_2(\xi_j|\mathbf{e})}$ when $\xi \in \operatorname{Supp}\chi$ (see eg [13, Lemma 2.9]). $\square$

We consider now the same problem for $\mathbb{B}^p_\nu$, namely the surjectivity of $\mathbb{B}^{p,(k)}_\mu \to \mathbb{B}^{p,(m)}_\mu$ for some $k < k_0(p,\mu)$. Again, this cannot happen at least if k is small.

Proposition 7.4.29. *Let $\mu \in \mathbb{R}$ and k be a non negative integer. If, for every $F \in \mathbb{B}^p_\mu$, there exists $\widetilde{F}$ such that $\Delta^k\Box^k\widetilde{F} \in L^p_\mu$ and $\Box^m\widetilde{F} = \Box^m F$ for some $m \ge k_0(p,\mu)$, then necessarily*

$$k + \tfrac{\mu}{p} \;>\; \max\Big\{ (\tfrac{n}{r}-1)\tfrac{1}{p}\,,\, (\tfrac{n}{r}-1)\,(\tfrac{1}{2}-\tfrac{1}{p}) \Big\}. \tag{7.4.21}$$

Proof. We must clearly have $\mu + kp > \frac{n}{r} - 1$, since otherwise $\Box^k\widetilde{F} \in A^p_{\mu+kp} = \{0\}$, which implies $F = 0 \pmod{\mathcal{N}_m}$. We may also assume that $k < k_0(p,\mu)$, since otherwise (7.4.21) is trivial. In particular, we only need to consider $p > 2$.

The proof is similar to Proposition 7.4.28 with some small changes. Under the condition in the statement, the inclusion $\mathbb{B}^{p,(k)}_\mu/\mathcal{N}_m \to \mathbb{B}^{p,(m)}_\mu$ is an isomorphism of Banach spaces. Hence, for every smooth f with Fourier transform compactly supported in Ω, the function $F = \mathcal{E}(f)$ belongs to $\mathbb{B}^{p,(m)}_\mu$ and there exists some $G \in \mathcal{H}(T_\Omega)$ with $\Box^m G = 0$ so that

$$\|\Delta^k\Box^k(F+G)\|_{L^p_\mu} \lesssim \|\Delta^m\Box^m F\|_{L^p_\mu}. \tag{7.4.22}$$

As before, $\Box^k G$ is the Fourier-Laplace transform of some distribution supported in $\partial\Omega$. Thus, for all $\widehat{\varphi} \in C^\infty_c(\Omega)$ we have

$$\begin{aligned}\Big|\int_{\mathbb{R}^n} \Box^k F(x+i\mathbf{e})\varphi(-x)\,dx\Big| \;&=\; \big|\Box^k(F+G)(\cdot+i\mathbf{e}) * \varphi(0)\big| \\ &\le\; \|\varphi\|_{p'}\,\big\|\Box^k(F+G)(\cdot+i\mathbf{e})\big\|_{L^p(\mathbb{R}^n)}. \end{aligned}\tag{7.4.23}$$

Since $\mu + kp > \frac{n}{r} - 1$ we have $\big\|\Box^k(F+G)(\cdot+i\mathbf{e})\big\|_{L^p(\mathbb{R}^n)} \lesssim \big\|\Box^k(F+G)\big\|_{A^p_{\mu+kp}(T_\Omega)}$ (see e.g. [13, Prop. 4.3]). By (7.4.22) and the results in Σ4.4, this last quantity is controlled by

$$\|\Box^m F\|_{A^p_{\mu+mp}} \lesssim \|\Box^m f\|_{B^p_{\mu+mp}} \approx \|f\|_{B^p_\mu},$$

since $m \geq k_0(p,\mu)$. Thus, going back to (7.4.23) we see that

$$\left|\int_{\mathbb{R}^n} f(x)\overline{T\varphi(x)}dx\right| \leq C\|f\|_{B^p_\mu} \times \|\varphi\|_{p'},$$

where as before $\widehat{T\varphi}(\xi) = \Delta^k(\xi)e^{-(\mathbf{e}|\xi)}\widehat{\varphi}(\xi)$. The left hand side can be written as a duality bracket $[f, T_\mu\varphi]_\mu$ by letting $\widehat{T_\mu\varphi}(\xi) = \Delta(\xi)^{k+\mu}e^{-(\mathbf{e}|\xi)}\widehat{\varphi}(\xi)$, and hence we conclude that

$$\|T_\mu\varphi\|_{B^{p'}_\mu} \leq C\|\varphi\|_{p'}\,. \tag{7.4.24}$$

As before, we choose $\varphi := \varphi_t$ with

$$\varphi_t(x) = \sum_{j\in J} r_j(t)a_j e^{i(x|\xi_j)}\eta(x),$$

where J is a finite set of indices and η is a smooth function with Fourier transform supported in a small ball centered at 0 so that ψ_j can be assumed to be equal to 1 on the support of $\hat{\eta}(\cdot - \xi_j)$, for all $j \in J$. Integrating in t and using Khintchine's inequality we find that

$$\int_0^1 \|T_\mu\varphi_t\|^{p'}_{B^{p'}_\mu}\, dt \leq C\int_0^1 \|\varphi_t\|^{p'}_{p'}\, dt \;\leq\; C'\left(\sum |a_j|^2\right)^{p'/2}\|\eta\|^{p'}_{p'}\,, \tag{7.4.25}$$

where the left hand side equals

$$\sum_{j\in J}\Delta(\xi_j)^{-\mu}|a_j|^{p'}\|T_\mu(e^{i(\cdot|\xi_j)}\eta)\|^{p'}_{p'}.$$

Arguing as in the proof of (7.4.19) one finds two constants c_1, c_2 such that

$$c_1\|T_\mu(e^{i(\cdot|\xi_j)}\eta)\|_{p'} \geq \Delta(\xi_j)^{k+\mu}e^{-c_2(\mathbf{e}|\xi_j)}\|\eta\|_{p'}.$$

So, (7.4.25) links to the existence of some constant C such that

$$\sum_{j\in J}|a_j|^{p'}\Delta(\xi_j)^{kp'+\mu p'-\mu}e^{-c_2(\mathbf{e}|\xi_j)} \leq C\left(\sum_{j\in J}|a_j|^2\right)^{p'/2}.$$

By the duality $\ell^r, \ell^{r'}$ with $r = 2/p'$ (since we assume $p > 2$), we conclude that

$$\sum_j \Delta(\xi_j)^{r'(kp'+\mu(p'-1))}e^{-c_3(\mathbf{e}|\xi_j)} < \infty,$$

since its partial sums are uniformly bounded. As in the previous proof, we conclude by a comparison with the corresponding integral, and find the constraint on parameters in (7.4.21).

□

Remark 7.4.30. In the special case $k = 0$ we obtain, for $\nu > \frac{n}{r} - 1$ and $m \geq k_0(p, \nu)$, that a necessary condition for the operator $\Box^m : A^p_\nu \to A^p_{\nu+mp}$ to be surjective is

$$1 \leq p < \frac{2(\nu + \frac{n}{r} - 1)}{\frac{n}{r} - 1} = \tilde{p}_\nu + \frac{\nu - 1}{\frac{n}{r} - 1}. \tag{7.4.26}$$

When $\nu \leq 1$ (in the three dimensional light-cone), (7.4.26) is the same necessary condition given in Conjecture 2. When $\nu > 1$, however, it is a weaker condition.

7.4.7 Complex interpolation

The Interpolation of Banach spaces is a powerful tool in Analysis. In this subsection we define and characterize the complex interpolation space of two Besov spaces. We first recall briefly the complex interpolation method.

The complex interpolation method

Two Banach spaces X_0 and X_1 are called compatible if there exists a Hausdorff topological linear space X containing both of them. In this case, we form two subspaces of X, $X_0 \cap X_1$ and $X_0 + X_1$, and they become Banach spaces with the following norms:

$$||x||_{X_0 \cap X_1} = \max(||x||_{X_0}, ||x||_{X_1}),$$

and

$$||x||_{X_0 + X_1} = \inf\{||x||_{X_0} + ||x||_{X_1} : x = x_0 + x_1 \, , x_0 \in X_0 \, , x_1 \in X_1\}.$$

Let $S = \{z \in \mathbb{C} : 0 < \Re z < 1\}$ denote the open strip and $\overline{S}$ its closure. If X_0 and X_1 are compatible Banach spaces, and if $\theta \in (0, 1)$, we define a Banach space X_θ as follows. As a vector space, X_θ consists of vectors $x \in X_0 + X_1$ with the following property: there exists a function $f : \overline{S} \to X_0 + X_1$ such that

(a) f is bounded and continuous on $\overline{S}$.

(b) f is analytic in S.

(c) $f(\theta) = x$

(d) $f(iy) \in X_0$ for every real y.

(e) $f(1 + iy) \in X_1$ for every real y.

For every f satisfying the above conditions we write

$$||f|| = \max\left(\sup_{y\in\mathbb{R}} ||f(iy)||_{X_0}, \sup_{y\in\mathbb{R}} ||f(1+iy)||_{X_1}\right).$$

The norm of $x \in X_\theta$ is then defined as the infimum of all such $||f||$.

To emphasize the dependence of X_θ on X_0 and X_1, we write

$$X_\theta = [X_0, X_1]_\theta,$$

and call it a complex interpolation space between X_0 and X_1. The construction of complex interpolation spaces is functorial in the following sense (see [19]).

Theorem 7.4.31. *Suppose X_0 and X_1 are compatible, Y_0 and Y_1 are compatible, and $\theta \in (0,1)$. If a linear operator $T : X_0 + X_1 \to Y_0 + Y_1$ maps X_0 boundedly into Y_0 (with norm M_0) and X_1 boundedly into Y_1 (with norm M_1), then T maps $[X_0, X_1]_\theta$ boundedly into $[Y_0, Y_1]_\theta$ (with norm not to exceed $M_0^{1-\theta} M_1^\theta$).*

One of the most important example of complex interpolation spaces is the following result concerning L^p spaces (see [19]).

Theorem 7.4.32. *If (X, μ) is a measurable space and $1 \leq p_0 < p_1 \leq \infty$, then*

$$[L^{p_0}(X), L^{p_1}(X)]_\theta = L^p(X)$$

with equal norms, where $0 < \theta < 1$ and

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Complex interpolation of analytic Besov spaces

We first consider the complex interpolation of two Bergman spaces. Following the complex method, if $\theta \in (0,1)$, we define the complex interpolation space $[A^{p_0}_{\nu_0}, A^{p_1}_{\nu_1}]_\theta$ of two Bergman spaces $A^{p_0}_{\nu_0}$ and $A^{p_1}_{\nu_1}$ as the space of holomorphic functions F in T_Ω such that there exists a function $z \mapsto f(z) = F_z$ from $\overline{S}$ into the Banach space $A^{p_0}_{\nu_0} + A^{p_1}_{\nu_1}$ such that properties (a) through (e) hold.

Theorem 7.4.33. *Suppose $\nu_0 > \frac{n}{r} - 1$ and $\nu_1 > \frac{n}{r} - 1$. If $1 \leq p_0 < \tilde{p}_{\nu_0}$, $1 \leq p_1 < \tilde{p}_{\nu_1}$, $1 \leq p_0 < p_1 < \infty$ and Hardy's inequality holds for both (p_0, ν_0) and (p_1, ν_1), and*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1},$$

for some $\theta \in (0,1)$, then

$$[A^{p_0}_{\nu_0}, A^{p_1}_{\nu_1}]_\theta = A^p_\nu$$

with equivalent norms, where ν is determined by

$$\frac{\nu}{p} = \frac{1-\theta}{p_0}\nu_0 + \frac{\theta}{p_1}\nu_1.$$

Proof. We naturally have $1 \leq p < \infty$. We first consider the case where $1 \leq p_0 < q_{\nu_0}$ and $1 \leq p_1 < q_{\nu_1}$, where $q_\alpha = 1 + \frac{\alpha}{\frac{n}{r}-1}$. Now we fix a real μ sufficiently large. By Corollary 6.3.9, the integral operator P_μ defined by

$$P_\mu f(z) = \int_{T_\Omega} B_\mu(z,w) f(w) \Delta^{\mu-\frac{n}{r}}(Imw)dw$$

maps L^p_ν boundedly onto A^p_ν, it maps $L^{p_0}_{\nu_0}$ boundedly onto $A^{p_0}_{\nu_0}$, and it maps $L^{p_1}_{\nu_1}$ boundedly onto $A^{p_1}_{\nu_1}$. It follows from the properties of complex interpolation and the settings above that P_μ maps $[L^{p_0}_{\nu_0}, L^{p_1}_{\nu_1}]_\theta = L^p_\nu$ boundedly onto $[A^{p_0}_{\nu_0}, A^{p_1}_{\nu_1}]_\theta$. Since $1 \leq p < q_\nu$, by Lemma 5.1 of [14], $P_\mu(L^p_\nu) = A^p_\nu$. We conclude that

$$A^p_\nu \subset [A^{p_0}_{\nu_0}, A^{p_1}_{\nu_1}]_\theta$$

and the inclusion is continuous.

Now if m is a positive integer, then the operator defined by

$$L(f)(z) = \Delta^m(\Im z)\Box^{(m)}_z f(z), \quad f \in \mathcal{H}(T_\Omega)$$

maps $A^{p_0}_{\nu_0}$ boundedly into $L^{p_0}_{\nu_0}$, it maps $A^{p_1}_{\nu_1}$ boundedly into $L^{p_1}_{\nu_1}$. It follows from the properties of complex interpolation and the same settings above that L maps $[A^{p_0}_{\nu_0}, A^{p_1}_{\nu_1}]_\theta$ boundedly into $[L^{p_0}_{\nu_0}, L^{p_1}_{\nu_1}]_\theta = L^p_\nu$. So, if $f \in [A^{p_0}_{\nu_0}, A^{p_1}_{\nu_1}]_\theta$, then, the function $z \mapsto \Delta^m(\Im z)\Box^m_z f(z)$ belongs to L^p_ν which because of $1 \leq p < q_\nu$ is equivalent to $f \in A^p_\nu$. We conclude that

$$[A^{p_0}_{\nu_0}, A^{p_1}_{\nu_1}]_\theta \subset A^p_\nu$$

and the inclusion is continuous. This completes the proof of the theorem when $1 \leq p_0 < q_{\nu_0}$ and $1 \leq p_1 < q_{\nu_1}$.

Next we consider the case where $2 < p_0 < p_1 < \infty$ and both P_{ν_0} and P_{ν_1} are bounded respectively on $L^{p_0}_{\nu_0}$ and $L^{p_1}_{\nu_1}$. For any integer m such that $p_0 < q_{\nu_0+mp_0}$ and $p_1 < q_{\nu_1+mp_1}$, using the fact that the $\Box^m$ is a bicontinuous isomorphism from $A^{p_j}_{\nu_j}$ onto $A^{p_j}_{\nu_j+mp_j}$, $j=0,1$ and the first part of the proof, we conclude that

$$[A^{p_0}_{\nu_0}, A^{p_1}_{\nu_1}]_\theta = A^p_\nu.$$

We conclude easily for the proof of theorem using the Wolff's abstract reiteration theorem (see [108]). □

We recall that $A^p_\nu = \mathcal{B}^p_\nu$, when $1 \leq p < \overline{p}_\nu$ and $\nu > \frac{n}{r} - 1$. Using the isomorphism of $\Box^m : \mathcal{B}^p_\nu \to \mathcal{B}^p_{\nu+mp}$ and the above theorem, we easily obtain the following.

Proposition 7.4.34. *Suppose both* (p_0, ν_0) *and* (p_1, ν_1) *are in* Υ. *If* $p_0 < p_1$ *and*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1},$$

for some $\theta \in (0,1)$, *then*

$$[\mathcal{B}^{p_0}_{\nu_0}, \mathcal{B}^{p_1}_{\nu_1}]_\theta = \mathcal{B}^p_\nu$$

with equivalent norms, where ν *is determined by*

$$\frac{\nu}{p} = \frac{1-\theta}{p_0}\nu_0 + \frac{\theta}{p_1}\nu_1.$$

Remark 7.4.35. The complex interpolate of two Bergman spaces with the same weight has been characterized in [16] using among other techniques, the Wolff's abstract reiteration theorem (see also [56]).

Let us now define the complex interpolate of two analytic Besov spaces. We recall the definition of

$$\mathbb{B}^{p,(m)}_\nu := \{F \in \mathcal{H}_m : \Box^m F \in A^p_{\nu+mp}\}.$$

Let us now introduce a notion of sum and intersection of Banach spaces (Besov spaces) of equivalence classes.

Let m be an integer such that $\nu_j + mp_j > \frac{n}{r} - 1$ and the Hardy's inequality holds for $(p_j, \nu_j + mp_j)$, $j = 0, 1$. We define the sum of two Besov spaces as follows:

$$\mathbb{B}^{p_0,(m)}_{\nu_0} + \mathbb{B}^{p_1,(m)}_{\nu_1} := \{F \in \mathcal{H}_m(T_\Omega) : \ \Box^m F \in A^{p_0}_{\nu_0+mp_0} + A^{p_1}_{\nu_1+mp_1}\}.$$

Endowed with the norm

$$||F||_{\mathbb{B}^{p_0,(m)}_{\nu_0} + \mathbb{B}^{p_1,(m)}_{\nu_1}} := ||\Box^m F||_{A^{p_0}_{\nu_0+mp_0} + A^{p_1}_{\nu_1+mp_1}},$$

$\mathcal{B}^{p_0,(m)}_{\nu_0} + \mathcal{B}^{p_1,(m)}_{\nu_1}$ is a Banach space. We also define their intersection as follows:

$$\mathbb{B}^{p_0,(m)}_{\nu_0} \bigcap \mathbb{B}^{p_1,(m)}_{\nu_1} := \{F \in \mathcal{H}_m(T_\Omega) : \ \Box^m F \in A^{p_0}_{\nu_0+mp_0} \bigcap A^{p_1}_{\nu_1+mp_1}\}.$$

Endowed with the norm

$$||F||_{\mathbb{B}^{p_0,(m)}_{\nu_0} \bigcap \mathbb{B}^{p_1,(m)}_{\nu_1}} := ||\Box^m F||_{A^{p_0}_{\nu_0+mp_0} \bigcap A^{p_1}_{\nu_1+mp_1}},$$

$\mathcal{B}^{p_0,(m)}_{\nu_0} \bigcap \mathcal{B}^{p_1,(m)}_{\nu_1}$ is a Banach space.

Definition 7.4.36. If $\theta \in (0,1)$, the complex interpolation space $[\mathbb{B}_{\nu_0}^{p_0,(m)}, \mathbb{B}_{\nu_1}^{p_1,(m)}]_\theta$ consists of functions $F \in \mathcal{H}_m(T_\Omega)$ such that there exists a function $z \mapsto f(z) = F_z$ from $\overline{S}$ into the Banach space $\mathbb{B}_{\nu_0}^{p_0,(m)} + \mathbb{B}_{\nu_1}^{p_1,(m)}$ so that properties (a) through (e) hold.

From the isomorphism property of the Box operator and the definition of $\mathbb{B}_\nu^{p,(m)}$, we clearly have that

$$\mathbb{B}_{\nu_0}^{p_0,(m)} \bigcap \mathbb{B}_{\nu_1}^{p_1,(m)} \subset [\mathbb{B}_{\nu_0}^{p_0,(m)}, \mathbb{B}_{\nu_1}^{p_1,(m)}]_\theta \subset \mathbb{B}_{\nu_0}^{p_0,(m)} + \mathbb{B}_{\nu_1}^{p_1,(m)}.$$

Moreover, we can always use the following equivalent norm for the interpolation space

$$||F||_{[\mathbb{B}_{\nu_0}^{p_0,(m)}, \mathbb{B}_{\nu_1}^{p_1,(m)}]_\theta} := ||\Box^m F||_{[A_{\nu_0+mp_0}^{p_0}, A_{\nu_1+mp_1}^{p_1}]_\theta}.$$

Theorem 7.4.37. *Let $\nu_0, \nu_1 \in \mathbb{R}$ and $1 \le p_0 < p_1 < \infty$. Let m be an integer so that $\nu_j + mp_j > \frac{n}{r} - 1$ and Hardy's inequality holds for $(p_j, \nu_j + mp_j)$, $j = 0, 1$. If, moreover,*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1},$$

for some $\theta \in (0,1)$, then

$$[\mathbb{B}_{\nu_0}^{p_0,(m)}, \mathbb{B}_{\nu_1}^{p_1,(m)}]_\theta = \mathbb{B}_\nu^{p,(m)}$$

with equivalent norms, where ν is determined by

$$\frac{\nu}{p} = \frac{1-\theta}{p_0}\nu_0 + \frac{\theta}{p_1}\nu_1.$$

Proof. We first remark that if Hardy's inequality holds for both (p_0, ν_0+mp_0) and (p_1, ν_1+mp_1), then it also holds for $(p, \nu+mp)$ where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$$

and

$$\frac{\nu}{p} = \frac{1-\theta}{p_0}\nu_0 + \frac{\theta}{p_1}\nu_1.$$

Clearly, the isomorphism of $\Box^m$ gives that for any $F \in [\mathbb{B}_{\nu_0}^{p_0,(m)}, \mathbb{B}_{\nu_1}^{p_1,(m)}]_\theta$,

$$\Box^m F \in [A_{\nu_0+mp_0}^{p_0}, A_{\nu_1+mp_1}^{p_1}]_\theta = A_{\nu+mp}^p.$$

Thus, by definition, $F \in \mathbb{B}_\nu^{p,(m)}$.

Conversely, if we denote by $\mathcal{I}$ the application which associates to every $F \in A_{\mu+mq}^q$ the equivalence class of all solutions of $\Box^m G = F$, then $\mathcal{I}$ maps $A_{\nu_0+mp_0}^{p_0}$ boundedly into $\mathbb{B}_{\nu_0}^{p_0,(m)}$, it maps $A_{\nu_1+mp_1}^{p_1}$ boundedly into $\mathbb{B}_{\nu_1}^{p_1,(m)}$. It follows from the properties of complex interpolation and the previous observations that $\mathcal{I}$ maps $[A_{\nu_0+mp_0}^{p_0}, A_{\nu_1+mp_1}^{p_1}]_\theta = A_{\nu+mp}^p$ boundedly into $[\mathbb{B}_{\nu_0}^{p_0,(m)}, \mathbb{B}_{\nu_1}^{p_1,(m)}]_\theta$. Thus, since by definition $\mathcal{I}(A_{\nu+mp}^p) = \mathbb{B}_\nu^{p,(m)}$, we conclude that $\mathbb{B}_\nu^{p,(m)} \subset [\mathbb{B}_{\nu_0}^{p_0,(m)}, \mathbb{B}_{\nu_1}^{p_1,(m)}]_\theta$. $\square$

As a consequence of the above theorem, we have that when $0 \leq k \leq m$ and $\nu_j + kp_j > \frac{n}{r} - 1$, $j = 0, 1$, the natural projection

$$\begin{array}{ccc} [\mathbb{B}^{p_0,(k)}_{\nu_0}, \mathbb{B}^{p_1,(k)}_{\nu_1}]_\theta & \longrightarrow & [\mathbb{B}^{p_0,(m)}_{\nu_0}, \mathbb{B}^{p_1,(m)}_{\nu_1}]_\theta \\ F + \mathcal{N}_k & \longmapsto & F + \mathcal{N}_m \end{array} \tag{7.4.27}$$

is an isomorphism of Banach spaces, provided Hardy's inequality (7.1.3) holds for the indices $(p_j, \nu_j + p_j k)$, $j = 0, 1$.

7.5 Open Questions

In this section we pose some questions left open in this topic, in addition to Conjecture 2. Most questions concern the spaces A^p_ν for $p \geq \tilde{p}_\nu$, about which we know very little.

(I) Is the operator $\Box^m : A^p_\nu \to A^p_{\nu+mp}$ onto for some $p \geq \tilde{p}_\nu$ and $m \geq k_0(p, \nu)$?

Equivalently, given a function $G \in A^p_{\nu+mp}$, does the equation

$$\Box^m F = G$$

have some solution F belonging to the space $A^p_\nu(T_\Omega)$?

From Remark 7.4.30 we only have a negative answer when $p \geq \tilde{p}_\nu + (\nu - 1)/(\frac{n}{r} - 1)$.

(II) Is the operator $\Phi : A^{q'}_\nu \to (A^q_\nu)^*$ onto for some $q \leq \tilde{p}'_\nu$?

This question is equivalent to (I) for $p = q'$, using the duality property $(A^q_\nu)^* = \mathbb{B}^p_\nu$ in Corollary 7.4.7.

(III) Is the Box operator injective on A^p_ν when $p = \tilde{p}_\nu$?

Injectivity holds when $1 \leq p < \tilde{p}_\nu$ (by Proposition 7.2.8), and fails when $p > \tilde{p}_\nu$ (by the explicit example $\Delta(z + i\mathbf{e})^{-\frac{n}{r}+1}$). We do not have a conjecture for the endpoint $p = \tilde{p}_\nu$.

(IV) Is the mapping $\Phi\colon A^{q'}_\nu \to (A^q_\nu)^*$ injective when $q = \tilde{p}'_\nu$?

This is equivalent to *(III)*. In fact, from (7.3.4) it easily seen that $\text{Ker}\ \Phi|_{A^{\tilde{p}_\nu}_\nu} = \text{Ker}\Box|_{A^{\tilde{p}_\nu}_\nu}$.

Our next question stresses further the differences between the spaces A^p_ν, depending on whether $p < \tilde{p}_\nu$ are $p \geq \tilde{p}_\nu$:

(V) Is the space A^p_ν isomorphic to ℓ^p for some $p \geq \tilde{p}_\nu$?

Recall here that the Bergman spaces A^p_ν are isomorphic to ℓ^p in the one dimensional setting. This can be proved as a consequence of the atomic decomposition (see [89]). In [10], atomic decompositions for A^p_ν are derived when Hardy's inequality holds.

(VI) Is it span $\{B_\mu(\cdot,w)\ :\ w\in T_\Omega\}$ dense in $A^p_\nu(T_\Omega)$ for $p\geq\tilde p_\nu$ and μ sufficiently large?

The validity of this result was wrongly stated in [14, Corollary 5.4] in the light-cone setting. As we show below (see also [14, Lemma 5.1]), the density holds when the projection P_μ is bounded in L^p_ν, but this restricts p to be smaller than $\tilde p_\nu$ (since $P^*_\mu=T_{\nu,\mu-\nu}$ must also be bounded in $L^{p'}_\nu$).

Proposition 7.5.1. *Let $\nu>\frac{n}{r}-1$. Assume that p and μ are so that P_μ extends as a bounded operator in L^p_ν. Then A^p_ν is the closed linear span of the set $\{B_\mu(\cdot,w),w\in T_\Omega\}$.*

Proof. The boundedness of P_μ in L^p_ν already implies that $B_\mu(\cdot,i\mathbf{e})\in A^p_\nu$. We take for granted the fact that $P^*_\mu=T_{\nu,\,\mu-\nu}$ (with respect to $\langle\cdot,\cdot\rangle_{dV_\nu}$). To establish the proposition it suffices to prove that for $f\in L^{p'}_\nu$ such that

$$\langle f,B_\mu(\cdot,w)\rangle_\nu=0\quad\forall\ w\in T_\Omega,\tag{7.5.1}$$

we have also $\langle f,F\rangle_\nu=0$ for all F in a dense subset of A^p_ν. Now (7.5.1) is the same as $T_{\nu,\,\mu-\nu}(f)(w)=0$, by definition of this operator. Thus, if $F\in A^p_\nu\cap A^2_\mu$, using the claim above we have

$$\langle f,F\rangle_\nu=\langle f,P_\mu F\rangle_\nu=\langle P^*_\mu(f),F\rangle_\nu=0.$$

Finally, we establish the claim, that is $P^*_\mu=T_{\nu,\,\mu-\nu}$. For $f,g\in C_c(T_\Omega)$ we have to justify the exchange of order of integration in

$$\begin{aligned}\langle P_\mu(g),f\rangle_\nu &= \int_{T_\Omega}\Big[\int_{T_\Omega}B_\mu(z,w)g(w)dV_\mu(w)\Big]\,\overline{f(z)}\,dV_\nu(z)\\ &= \int_{T_\Omega}g(w)\Big[\int_{T_\Omega}\overline{B_\mu(w,z)f(z)}dV_\nu(z)\Big]\,dV_\mu(w)\ =\ \langle g,T_{\nu,\,\mu-\nu}f\rangle_\nu.\end{aligned}$$

but this follows from

$$\int_{T_\Omega}\int_{T_\Omega}|B_\mu(z,w)||g(w)|\,dV_\mu(w)|f(z)|\,dV_\nu(z)\leq\big\|T^+_{\mu,0}|g|\big\|_{L^2_\mu}\big\|\Delta^{\nu-\mu}|f|\big\|_{L^2_\mu}<\infty,$$

using the fact that the operator $T^+_{\mu,0}$ with kernel $|B_\mu(z,w)|$ is bounded on L^2_μ. ∎

Chapter 8

Hankel operators on Bergman spaces

We present here some criteria for Schatten-Von Neumann class membership for the small Hankel operator on the Bergman space $A^2_\nu(T_\Omega)$, when $T_\Omega = \mathbb{R}^n + i\Omega$ is the tube over the symmetric cone Ω. For simplicity, we restrict ourself to the unweighted case since the general result is proved in the same way.

8.1 Introduction

Let Ω be an irreducible symmetric cone in the Euclidean vector space $\mathbb{R}^n$ of dimension n, endowed with an inner product $(\cdot|\cdot)$ for which the cone Ω is self-dual. We denote by $T_\Omega = \mathbb{R}^n + i\Omega$ the corresponding tube domain in $\mathbb{C}^n$. Again, we write the rank and determinant associated with the cone by

$$r = rank\ \Omega, \text{ and } \Delta(x) = det\, x,\ x \in \mathbb{R}^n.$$

For more simplicity, we modify our definition of Bergman space by translating the weight. This has no effect on the results. Given $1 \leq p < \infty$ and $\nu > 2\frac{n}{r} - 1$, the weighted Bergman space $A^p_\nu(T_\Omega)$ of the tube T_Ω is the space of analytic functions f on T_Ω satisfying the integrability condition

$$||f||_{A^p_\nu} := \left(\int_{T_\Omega} |f(x+iy)|^p \Delta^{\nu - 2\frac{n}{r}}(y)dxdy\right)^{\frac{1}{p}} < \infty. \tag{8.1.1}$$

When $\nu = 2\frac{n}{r}$, we write $A^2(T_\Omega) = A^2_{2\frac{n}{r}}(T_\Omega)$.

For $1 \leq p < \infty$, the Besov space $\mathbb{B}^p(T_\Omega)$ of the tube T_Ω is the space of holomorphic functions $f \in \mathcal{H}_n(T_\Omega)$ such that

$$\int_{T_\Omega} |\Box^n f(x+iy)|^p \Delta(y)^{np-2n/r} \mathrm{d}x\mathrm{d}y < \infty.$$

In other words, f belongs to $\mathbb{B}^p(T_\Omega)$ if and only if $\Box^n f$ belongs to $A^p_{np}(T_\Omega)$. When $p = \infty$, we denote the Bloch space of T_Ω by $\mathbb{B} = \mathbb{B}^\infty$, which is the space of analytic functions f satisfying

$$\sup_{z \in T_\Omega} \Delta^n(\Im z)|\Box^n f(z)| < \infty.$$

Remark 8.1.1. In our notations in the previous chapter, the $\mathbb{B}^p$ here corresponds to $\mathbb{B}^{p,(n)}$ in the previous chapter while $\mathbb{B}$ corresponds to $\mathbb{B}^{\infty,(n)}$.

The weighted Bergman projection P_ν is given by the integral formula

$$P_\nu f(z) = \int_{T_\Omega} B_\nu(z,w) f(w) \Delta^{\nu - 2\frac{n}{r}}(\Im w) dV(w) \tag{8.1.2}$$

where

$$B_\nu(z,w) = d_\nu \Delta^{-\nu}\left(\frac{z - \overline{w}}{i}\right) \tag{8.1.3}$$

is the weighted Bergman kernel (see [12]) and dV is the Lebesgue measure on T_Ω. Let us recall that B_ν is a reproducing kernel on $A^2_\nu(T_\Omega)$, that is for every $f \in A^2_\nu(T_\Omega)$ we have the formula:

$$f(z) = c \int_{T_\Omega} B_\nu(z,w) f(w) \Delta^{\nu - 2\frac{n}{r}}(\Im w) dV(w). \tag{8.1.4}$$

In fact, formula (7.2.13) gives that for any $\mu > 2\frac{n}{r} - 1$, and any $f \in A^2_\nu(T_\Omega)$ we still have

$$f(z) = c \int_{T_\Omega} B_\mu(z,w) f(w) \Delta^{\mu - 2\frac{n}{r}}(\Im w) dV(w). \tag{8.1.5}$$

Again, when $\nu = 2\frac{n}{r}$, we write $P = P_\nu$ and $B = B_\nu$.

Let $b \in L^2(T_\Omega) = L^2(T_\Omega, dV)$. The small Hankel operator h_b with symbol b is defined as

$$h_b(f) = P(b\overline{f}) \tag{8.1.6}$$

for $f \in H^\infty(T_\Omega)$.

The aim of this chapter is to give criteria for Schatten class ($\mathcal{S}_p$) membership of Hankel operators on the Bergman space $A^2(T_\Omega)$. This problem has been considered in [2], [65] for the case of the unit disc of the complex plane, and in [116] and [115] for bounded symmetric domains. Some earlier works were done in [1], [35], [61], [80] and [90] in various domains including the upper half plane. It is shown in those cases that the small Hankel operator is in the Schatten class $\mathcal{S}_p$ if and only if its symbol belongs to the corresponding Besov space $\mathbb{B}^p$. The idea of the proof in [115] is the use of an appropriate integral operator which carries a lot of information on the small Hankel operator. This idea seems to be the appropriate one in our case also. Let us mention that the same problem for Hardy space of tube domains over symmetric cones was considered in [24] where it is stated that classical result extends to this case at least for $1 \leq p \leq 2$.The main tool in the proof of the necessity in [24] is the use of the sampling theorem related to a lattice in T_Ω for functions in a Bergman space. We will also take advantage of this idea. We show specially that classical results (see [115] for example) extend to the tube domains over symmetric cones for the range $1 \leq p \leq \infty$. When the symbol is analytic and $1 \leq p \leq \infty$, we also obtain criteria in terms of the action of the operator on the reproducing kernel, here, "the reproducing kernel thesis". This last characterization appears in [98] for the same problem in the case of Hardy space of the unit disc. The main result of this chapter can be stated in the following way.

Theorem 8.1.2. *Suppose b is analytic in T_Ω and $1 \leq p \leq \infty$. Then the following conditions are equivalent*

(i) $h_b \in \mathcal{S}_p$.

(ii) $b \in \mathbb{B}^p$.

(iii) *For all integer* $k > \frac{n}{r} - 1$,

$$\int_{T_\Omega} ||h_b(\Box^k B)(., z)||^p \Delta^{(k+n/r)p-2n/r}(\Im z) dV(z) < \infty.$$

Here the norm $||\ ||$ is the norm of the Hilbert space on which the operator acts, that is $A^2(T_\Omega)$.

Remark 8.1.3.

- Condition *(iii)* is still valid when b is not analytic since it only depends on the operator. One can weaken the conditions on k. We will see, in particular, that it is sufficient to consider $k \geq 0$ when $p \geq 2$.
- The operator as defined above does not depend on the choice of the representative of a class. This is an easy consequence of the formula (7.2.8).
- The choice of the unweighted case is only for simplicity of our presentation. The results easily generalize to the weighted case.

8.2 Preliminaries

8.2.1 Determinant function, Schatten-Von Neumann classes

We adopt the following notation

$$B_z(.) = B(., z) = d\Delta^{-2\frac{n}{r}}(\frac{. - \overline{z}}{i}).$$

We recall that $A^2(T_\Omega)$ is a Hilbert space with reproducing kernel B_z, that is for every $f \in A^2(T_\Omega)$, $\langle f, B_z\rangle = f(z)$ with the pairing

$$\langle f, g\rangle = \int_{T_\Omega} f(z)\overline{g(z)}dV(z).$$

With our new notations, Lemma 6.3.2 (for $p = q$) takes the following form.

Lemma 8.2.1. *Let α be real.Then the function $f(z) = \Delta^{-\alpha}(\frac{z+it}{i})$, with $t \in \Omega$, belongs to $A^p_\nu(T_\Omega)$ if and only if*

$$\alpha > \max(\frac{2\frac{n}{r} - 1}{p}, \frac{\nu + \frac{n}{r} - 1}{p})$$

In this case,

$$||f||_{A^p_\nu} = C_{\alpha,p}\Delta^{-p\alpha+\nu}(t).$$

It follows easily that $||B_z||_{A^2} = C\Delta^{-(\frac{n}{r})}(\Im z)$.

Let us recall that the wave operator acts on the reproducing kernel in the following way:

$$\Box^k_{\overline{z}} B_\alpha(., z) = C_{n,\alpha,k}B_{\alpha+k}(., z)$$

with B_α given by the formula (8.1.3). As in the previous chapter, we have that for any $f \in A^2(T_\Omega)$,

$$\langle f, \Box^m_{\overline{z}} B_z\rangle = C_m\Box^m f(z).$$

In particular,

$$||\Box^m_{\overline{z}} B_z||_{A^2} = |\langle\Box^m_{\overline{z}} B_z, \Box^m_{\overline{z}} B_z\rangle|^{1/2} = C_m\Delta^{-(m+\frac{n}{r})}(\Im z).$$

For any $f \in A^2(T_\Omega)$ $(f \neq 0)$, we denote by $\tilde{f}$, the normalization of f, that is

$$\tilde{f} = f/||f||_{A^2(T_\Omega)}. \qquad (8.2.1)$$

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces. Let $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $\mathcal{K}(\mathcal{H}_1, \mathcal{H}_2)$ denote the spaces of bounded and compact linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$, respectively. It is well known (see for example [114]) that any operator $T \in \mathcal{K}(\mathcal{H}_1, \mathcal{H}_2)$ has a Schmidt decomposition, that is there exist orthonormal bases $\{e_j\}$ and $\{\sigma_j\}$ of $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively and a sequence $\{\lambda_j\}$ of complex numbers with $\lambda_j \to 0$, such that

$$Tf = \sum_{j=0}^{\infty} \lambda_j \langle f, e_j \rangle \sigma_j, \qquad f \in \mathcal{H}_1. \tag{8.2.2}$$

For $1 \leq p < \infty$, a compact operator T with such a decomposition belongs to the Schatten-Von Neumann p-class $\mathcal{S}_p(\mathcal{H}_1, \mathcal{H}_2)$, if and only if

$$||T||_{\mathcal{S}_p} = (\sum_{j=0}^{\infty} |\lambda_j|^p)^{\frac{1}{p}} < \infty.$$

When $T \in \mathcal{S}_p = \mathcal{S}_p(\mathcal{H}, \mathcal{H})$, for any image $\{e_j\}$ of an orthonormal sequence by a bounded operator in $\mathcal{H}$,

$$\sum_{j=0}^{\infty} |\langle Te_j, e_j \rangle|^p \lesssim ||T||_{\mathcal{S}_p}^p$$

(see [91]). For $p = 1$, $\mathcal{S}_1 = \mathcal{S}_1(\mathcal{H}, \mathcal{H})$ is the trace class and for $T \in \mathcal{S}_1$, the trace of T is defined by

$$Tr(T) = \sum_{j=0}^{\infty} \langle Te_j, e_j \rangle$$

where $\{e_j\}$ is any orthonormal basis of the Hilbert space $\mathcal{H}$. We will denote by $\mathcal{S}_\infty(\mathcal{H})$ the set of all bounded linear operators on $\mathcal{H}$.

8.2.2 Bergman distance, Sampling and Covering results

We first recall the definition of the Bergman distance on the tube T_Ω. Let $\{g_{j,k}\}_{1 \leq j,k \leq n}$ be the matrix function defined on T_Ω by

$$g_{j,k}(z) = \frac{\partial^2}{\partial z_j \partial \overline{z}_k} \log B(z, z).$$

The mapping $z \in T_\Omega \mapsto \mathcal{H}_z$ where

$$\mathcal{H}_z(u, v) = \sum_{1 \leq j,k \leq n} g_{j,k}(z) u_j \overline{v}_k \quad (u = (u_1, \cdots, u_n), \;\; v = (v_1, \cdots, v_n) \in \mathbb{C}^n),$$

defines a Hermitian metric on $\mathbb{C}^n$, called the Bergman metric. The length of a smooth path $\gamma : [0, 1] \to T_\Omega$ is given by

$$\ell(\gamma) = \int_0^1 \{\mathcal{H}_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))\}^{\frac{1}{2}} dt$$

and the Bergman distance between two points z_1, z_2 of T_Ω is

$$d_{Berg}(z_1, z_2) = \inf_{\gamma} \ell(\gamma)$$

where the infinimum is taken over all smooth paths $\gamma : [0,1] \to T_\Omega$ such that $\gamma(0) = z_1$ and $\gamma(1) = z_2$.

Remark 8.2.2. We refer to the text [58] for the following observations:

(a) The Bergman distance d_{Berg} is equivalent to the Euclidean distance on the compact sets of $\mathbb{C}^n$ contained in T_Ω.

(b) The Bergman balls in T_Ω are relatively compact.

(c) Let $\mathbb{R}^n$ be the group of translations by vectors and let again H be the simply transitive group of automorphisms of the symmetric cone Ω defined in Chapter 5. The group $\mathbb{R}^n \times H$ acts simply transitively on T_Ω and the Bergman distance is invariant under the automorphisms of $\mathbb{R}^n \times H$.

Let us denote by $B_\eta(z)$ the Bergman ball centered at z with radius η. We have the following covering lemma in [24].

Lemma 8.2.3. *Given $\delta \in (0,1)$, there exists a sequence of points $\{z_j\}$ in T_Ω called δ-lattice such that, calling B_j and B'_j the Bergman balls with center z_j and radius δ and $\delta/2$ respectively, then*

(i) the balls B'_j are pairwise disjoint;

(ii) the balls B_j cover T_Ω with finite overlapping, i.e. there is an integer N such that each point of T_Ω belongs to at most N of these balls.

We observe with [12] the following.

Lemma 8.2.4. *For all $\delta \in]0,1]$, there exists a constant $C > 1$ such that if $d_{Berg}(z,w) < \delta$ for some $z, w \in T_\Omega$, then $\frac{1}{C} < \frac{\Delta(\Im z)}{\Delta(\Im w)} < C$.*

The above balls have the following properties:

$$\int_{B_j} dV(z) \approx \int_{B'_j} dV(z) \approx C_\delta \Delta^{2n/r}(\Im z_j).$$

We recall that the measure $d\lambda(z) = \Delta^{-2n/r}(\Im z)dV(z)$ is an invariant measure under the automorphism of T_Ω ([12]).

The proof of the following sampling result heavily uses Lemma 8.2.4.

Lemma 8.2.5. *(Theorem 5.6 in [12]) Let $\{z_j\}$ be a δ-lattice in T_Ω, $\delta \in (0,1)$.*

(i) There exists a positive constant C_δ such that every $f \in A^p_\nu(T_\Omega)$ satisfies

$$\sum_j |f(z_j)|^p \Delta^\nu(\Im z_j) \leq C_\delta ||f||^p_{A^p_\nu}.$$

(ii) Conversely, if δ is small, there is a positive constant C_δ such that every $f \in A^p_\nu(T_\Omega)$ satisfies

$$||f||^p_{A^p_\nu} \leq C_\delta \sum_j |f(z_j)|^p \Delta^\nu(\Im z_j).$$

8.3 $\mathcal{S}_p$ criteria for arbitrary operators on $A^2(T_\Omega)$

8.3.1 Hilbert-Schmidt operators

These are operators in $\mathcal{S}_2$. The following result is established for an arbitrary operator defined on $A^2(T_\Omega)$ with values in a Hilbert space $\mathcal{H}$.

Theorem 8.3.1. *If $T \in \mathcal{B}(A^2(T_\Omega), \mathcal{H})$ then*

$$||T||^2_{\mathcal{S}_2(A^2(T_\Omega),\mathcal{H})} = C_{n,k} \int_{T_\Omega} ||T(\widetilde{\Box^k_{\bar{z}} B_z})||^2 d\lambda(z),$$

for every integer $k \geq 0$, where $d\lambda(z) = \Delta^{-2\frac{n}{r}}(\Im z) dV(z)$ is the invariant measure on T_Ω .

Proof. If $\{e_j\}$ is an orthonormal basis of $\mathcal{H}$, then

$$\begin{aligned}
\int_{T_\Omega} ||T(\Box^k_{\bar{z}} B_z)||^2 \Delta^{2k}(\Im z) dV(z) &= \int_{T_\Omega} \sum_{j=0}^{\infty} |\langle T(\Box^k_{\bar{z}} B_z), e_j\rangle|^2 \Delta^{2k}(\Im z) dV(z) \\
&= \sum_{j=0}^{\infty} \int_{T_\Omega} |\langle \Box^k_{\bar{z}} B_z, T^* e_j\rangle|^2 \Delta^{2k}(\Im z) dV(z) \\
&= \sum_{j=0}^{\infty} \int_{T_\Omega} |\Box^k T^* e_j(z)|^2 \Delta^{2k}(\Im z) dV(z) \\
&= C_{n,k} \sum_{j=0}^{\infty} \int_{T_\Omega} |T^* e_j(z)|^2 dV(z) \\
&= C_{n,k} \sum_{j=0}^{\infty} ||T^* e_j||^2_{A^2} \\
&= C_{n,k} ||T^*||^2_{\mathcal{S}_2} = C_{n,k} ||T||^2_{\mathcal{S}_2}.
\end{aligned}$$

The fourth equality follows from the fact that $\Box^k$ is an isometric (up to constant $C_{n,k}$) isomorphism from $A^2(T_\Omega)$ onto $A^2_{2k+2n/r}(T_\Omega)$. $\Box$

8.3.2 $\mathcal{S}_p(A^2(T_\Omega),\mathcal{H})$ for $p\neq 2$

Lemma 8.3.2. *Suppose that $T\in\mathcal{B}(A^2(T_\Omega),\mathcal{H})$ for any Hilbert space $\mathcal{H}$ and $k=0,1,\cdots$. Then,*

i) if $T\in\mathcal{S}_p$ for $2<p<\infty$ then,

$$\int_{T_\Omega}||T(\widetilde{\Box_{\frac{k}{z}}B_z})||^p d\lambda(z)\leq C_{n,k}||T||^p_{\mathcal{S}_p}.$$

ii) If for $1\leq p<2$,

$$\int_{T_\Omega}||T(\widetilde{\Box_{\frac{k}{z}}B_z})||^p d\lambda(z)<\infty$$

then $T\in\mathcal{S}_p$. Moreover,

$$||T||^p_{\mathcal{S}_p}\leq C_{n,k}\int_{T_\Omega}||T(\widetilde{\Box_{\frac{k}{z}}B_z})||^p d\lambda(z).$$

Proof. First of all, we have by Theorem 8.3.1 that if $T\in\mathcal{S}_1(A^2(T_\Omega),A^2(T_\Omega))$ is a positive operator, then

$$Tr(T)=||T^{1/2}||^2_{\mathcal{S}_2}=C_{n,k}\int_{T_\Omega}\langle T(\Box_{\frac{k}{z}}B_z),\Box_{\frac{k}{z}}B_z\rangle\Delta^{2k}(\Im z)dV(z).$$

The result follows since $||T||^p_{\mathcal{S}_p}=Tr((T^*T)^{p/2})$ and for any unit vector(see [114]) in $L^2(\mathcal{D})$, we have

$$\langle T^*Tf,f\rangle^{p/2}\leq\ \langle (T^*T)^{p/2}f,f\rangle,\quad\text{if}\quad p>2$$

and

$$\langle (T^*T)^{p/2}f,f\rangle\leq\ \langle T^*Tf,f\rangle^{p/2}\quad\text{if}\quad 1\leq p\leq 2.$$

□

8.4 The case of the small Hankel operators

We give in this section some Schatten classes membership criteria for the small Hankel operator on the Bergman space $A^2(T_\Omega)$. Let V_k and L be the operators defined on $L^2(T_\Omega)$ by

$$V_kf(z)=\Delta^{2k+2\frac{n}{r}}(\Im z)\int_{T_\Omega}B_{(2k+4\frac{n}{r})}(z,w)f(w)dV(w),\ \ z\in T_\Omega,$$

and

$$Lf(z)=\Delta^{n}(\Im z)\int_{T_\Omega}B_{n+2\frac{n}{r}}(z,w)f(w)dV(w),\ \ z\in T_\Omega.$$

We set $\tau_z=B_{(\frac{n}{2}+\frac{n}{r})}(.,z)$. We have the following lemma.

Lemma 8.4.1. *Let $\langle , \rangle$ be the inner product in $L^2(T_\Omega)$. For $f \in L^2(T_\Omega)$, we have*

1) $V_k f(z) = C_n \langle h_f(\widetilde{\Box_{\bar{z}}^k B_z}), \widetilde{\Box_{\bar{z}}^k B_z} \rangle$ *and* $Lf(z) = C_n \langle h_f \tilde{\tau}_z, \tilde{\tau}_z \rangle$.

2) $PV_k f = PLf = Pf$.

3) $h_f = h_{Pf} = h_{V_k f} = h_{Lf}$ *on* $A^2(T_\Omega)$.

Proof. 1) follows from the definition of V_k and L, Fubini's theorem and reproducing formulas.

Let us show 2). It is not hard to see that the operators V_k and L are bounded on $L^2(T_\Omega)$ as it follows from Theorem 6.3.3. We consider the following function:

$$F_z(\xi, w) = \Delta^{-2n/r}(\frac{z - \overline{\xi}}{i})\Delta^{-(n+2n/r)}(\frac{\xi - \overline{w}}{i})f(w)\Delta^n(\Im \xi), \quad f \in L^2(T_\Omega),$$

and $z \in T_\Omega$. Using the reproducing formula, we obtain that

$$\int_{T_\Omega} F(\xi, w)dV(w) = \Delta^{-2n/r}(\frac{z - \overline{\xi}}{i})Lf(\xi) := G_z(\xi),$$

and

$$\int_{T_\Omega} F(\xi, w)dV(\xi) = \Delta^{-2n/r}(\frac{z - \overline{w}}{i})f(w) := H_z(w).$$

It is clear that H_z is integrable and so is G_z since the operator L is bounded on $L^2(T_\Omega)$. Applying Fubini's theorem, we obtain

$$PLf(z) = \int_{T_\Omega} G_z(\xi)dV(\xi) = \int_{T_\Omega} H_z(w)dV(w) = Pf(z).$$

The equality $PV_k f = Pf$ follows in the same way. The first equality in 3) follows from the definition of the little Hankel operator, Fubini's theorem and reproducing formulas, the second and the third equalities follow from the first one and 2). $\Box$

Lemma 8.4.2. *If $1 \leq p \leq \infty$ and $b \in L^p(T_\Omega, d\lambda)$, then the Hankel operator h_b is in the Schatten class $\mathcal{S}_p$.*

Proof. The case $p = \infty$ is obvious, it suffices then to show the case $p = 1$ since the result then follows by interpolation. An easy computation shows that

$$h_b = \int_{T_\Omega} b(w)h_{f_w}d\lambda(w),$$

where $f_w(z) = \Delta^{2\frac{n}{r}}(\Im z)\Delta^{2\frac{n}{r}}(\Im w)\Delta^{-4\frac{n}{r}}(\frac{z-\overline{w}}{i})$ and h_{f_w} is the rank 1 Hankel operator given by

$$h_{f_w}g = \Delta^{2\frac{n}{r}}(\Im w)\Delta^{-2\frac{n}{r}}(\frac{\cdot - \overline{w}}{i})\overline{g(w)}$$

with $||h_{f_w}||_{\mathcal{S}_1} = ||h_{f_w}|| = C < \infty$. It follows that

$$||h_b||_{\mathcal{S}_1} \leq \int_{T_\Omega} ||h_{f_w}||_{\mathcal{S}_1} |b(w)| d\lambda(w) \leq C \int_{T_\Omega} |b(w)| d\lambda(w).$$

The proof is complete. □

Theorem 8.4.3. *Suppose $1 \leq p \leq \infty$, and $b \in L^2(T_\Omega)$. Then the following assertions are equivalent*

i) h_b is in $\mathcal{S}_p$.

ii) For every integer $k \geq 0$, $V_k b \in L^p(T_\Omega, d\lambda)$.

Proof. *ii)* $\Rightarrow$ *i)* follows from Lemma 8.4.2 and the equality $h_b = h_{V_k b}$. Let us show that for

$1 \leq p < \infty$, *i)* $\Rightarrow$ *ii)*. Let $\{z_j\}$ be a δ-lattice in T_Ω. Using the equality
$V_k b(z) = C_{n,k} \langle h_b(\widetilde{\Box^k_{\bar{z}} B_z}), \widetilde{\Box^k_{\bar{z}} B_z} \rangle$ and Lemma 8.2.5, we obtain

$$\begin{aligned}
||V_k b||^p_{L^p(T_\Omega, d\lambda)} &= C_{n,k} \int_{T_\Omega} |\langle h_b(\widetilde{\Box^k_{\bar{z}} B_z}), \widetilde{\Box^k_{\bar{z}} B_z} \rangle|^p d\lambda(z) \\
&= C_{n,k} \int_{T_\Omega} |\langle h_b(\Box^k_{\bar{z}} B_z), \Box^k_{\bar{z}} B_z \rangle|^p \Delta^{2k}(\Im z) dV(z) \\
&\approx C_{n,k} \sum_j |\langle h_b(\Box^k_{\bar{z}} B_{z_j}), \Box^k_{\bar{z}} B_{z_j} \rangle|^p \Delta^{2k+2\frac{n}{r}}(\Im z_j) \\
&= C_{n,k} \sum_j |\langle h_b(\widetilde{\Box^k_{\bar{z}} B_{z_j}}), \widetilde{\Box^k_{\bar{z}} B_{z_j}} \rangle|^p
\end{aligned}$$

.

To conclude, it suffices to show that $\widetilde{\Box^k_{\bar{z}} B_{z_j}} = C_k \Delta^{k+n/r}(\Im z_j) \Delta^{-(k+2n/r)}(\frac{\cdot - \bar{z}_j}{i})$ is the image of an orthonormal sequence ψ_j in $L^2(T_\Omega)$ through a bounded linear map $T_k : L^2(T_\Omega) \mapsto L^2(T_\Omega)$.

Define $T_k : L^2(T_\Omega) \mapsto L^2(T_\Omega)$ by setting

$$T_k \psi(z) = C_{n,k} \int_{T_\Omega} \Delta^{-(k+2\frac{n}{r})}(z - \bar{\xi}) \psi(\xi) \Delta^k(\Im \xi) dV(\xi), \quad z \in T_\Omega$$

and $\psi_j(z) = C_{n,k} \Delta^{-n/r}(\Im z) \chi_{B'_j}(z)$. Then $T_k \psi_j = \widetilde{\Box^k_{\bar{z}} B_{z_j}}$, $||\psi_j||_{L^2(T_\Omega)} = 1$ with an appropriate choice of $C_{n,k}$. The operator $T_k = C_k P_{k+2\frac{n}{r}}$ is clearly bounded on $L^2(T_\Omega)$ by Corollary 6.3.9.

For $p = \infty$, we take as test functions $f_z = g_z = \widetilde{\Box^k_{\bar{z}} B_z}$. It follows that if h_b is bounded on $A^2(T_\Omega)$, then

$$|V_k b(z)| = C_{n,k} |\langle h_b(\widetilde{\Box^k_{\bar{z}} B_z}), \widetilde{\Box^k_{\bar{z}} B_z} \rangle| = C_{n,k} |\langle h_b(f_z), g_z \rangle| < \infty.$$

So $V_k b \in L^p(T_\Omega, d\lambda)$ for any $1 \leq p \leq \infty$. The proof is complete. □

Theorem 8.4.4. *Suppose $1 \leq p \leq \infty$, and $b \in L^2(T_\Omega)$ is analytic. Then the following assertions are equivalent*

i) h_b *is in* $\mathcal{S}_p$.

ii) $b \in \mathbb{B}^p$.

Proof. *ii)* $\Rightarrow$ *i)* follows from Lemma 8.4.2 and the equalities $Lb(z) = \Delta^n(\Im z)\Box^n b(z)$ and $h_b = h_{Lb}$. Let show that *i)* $\Rightarrow$ *ii)* for $1 \leq p < \infty$. Let $\{z_j\}$ be a δ-lattice in $\mathcal{D}$. Using the equality $Lb(z) = C_n\langle h_b\tilde{\tau}_z, \tilde{\tau}_z\rangle$ and Lemma 8.2.5, we have

$$\begin{aligned}
||\Box^n b||^p_{A^p_{np}} = ||Lb||^p_{L^p(\mathcal{D},d\lambda)} &= C_n\int_{T_\Omega}|\langle h_b\tilde{\tau}_z, \tilde{\tau}_z\rangle|^p d\lambda(z)\\
&= C_n\sum_j\int_{B_j}|\langle h_b(\tau_z), \tau_z\rangle|^p\Delta^{np-2\frac{n}{r}}(\Im z)dV(z)\\
&\approx C\sum_j|\langle h_b(\tau_{z_j}), \tau_{z_j}\rangle|^p\Delta^{np}(\Im z_j)\\
&= C\sum_j|\langle h_b\tilde{\tau}_{z_j}, \tilde{\tau}_{z_j}\rangle|^p.
\end{aligned}$$

To conclude, it suffices to show that $\tilde{\tau}_{z_j}$ is the image of an orthonormal sequence φ_j in $L^2(T_\Omega)$ trough a bounded linear map $T : L^2(T_\Omega) \mapsto L^2(T_\Omega)$.

Define $T : L^2(T_\Omega) \mapsto L^2(T_\Omega)$ by setting

$$T\varphi(z) = C_n\int_{T_\Omega}\Delta^{-(\frac{n}{2}+\frac{n}{r})}(z-\overline{\xi})\varphi(\xi)\Delta^{(\frac{n}{2}-\frac{n}{r})}(\Im\xi)dV(\xi), \quad z \in T_\Omega$$

and $\varphi_j(z) = C_n\Delta^{-n/r}(\Im z)\chi_{B'_j}(z)$. Then $T\varphi_j = \tilde{\tau}_{z_j}$, $||\varphi_j||_{L^2(T_\Omega)} = 1$ with an appropriate choice of C_n. The operator $T = C_nP_{(\frac{n}{2}+\frac{n}{r})}$ is bounded on $L^2(T_\Omega)$ by Corollary 6.3.9. The case $p = \infty$ can be handled easily as in Theorem 8.4.3 taking as test functions $f_z = g_z = \tilde{\tau}_z$. The proof is complete. $\Box$

8.5 The reproducing kernel thesis

In this section, we give Schatten class criteria for the little Hankel operator on the Bergman space $A^2(T_\Omega)$ in terms of the action of the operator on the reproducing kernel. We refer the reader to [6, 25, 57] where some previous works have been considered.

Theorem 8.5.1. *Let $b \in A^2(T_\Omega)$. Then the following conditions are equivalent*

i) h_b *is bounded on* $A^2(T_\Omega)$.

ii) For every integer $k \geq 0$,

$$\sup_{z \in T_\Omega} ||h_b(\widetilde{\Box_{\bar{z}}^k B_z})||_{A^2(T_\Omega)} < \infty.$$

Proof. That $i) \Rightarrow ii)$ is obvious. Let show that $ii) \Rightarrow i)$. For that, it suffices by Theorem 8.4.3 to show that ii) implies that $\sup_{z \in T_\Omega} |V_k b(z)| < \infty$. But, we already know that

$$V_k b(z) = C_{n,k} \langle h_b(\widetilde{\Box_{\bar{z}}^k B_z}), \widetilde{\Box_{\bar{z}}^k B_z} \rangle.$$

The result follows now since

$$|V_k b(z)| \leq C_{n,k} ||h_b(\widetilde{\Box_{\bar{z}}^k B_z})||_{A^2} ||\widetilde{\Box_{\bar{z}}^k B_z}||_{A^2} = C_{n,k} ||h_b(\widetilde{\Box_{\bar{z}}^k B_z})||_{A^2}.$$

The proof is complete. □

Theorem 8.5.2. *Let $b \in L^2(T_\Omega)$ be analytic and $1 \leq p < \infty$. The following conditions are equivalent*

i) $h_b \in \mathcal{S}_p$.

ii) For every integer $k > \frac{n}{r} - 1$,

$$\int_{T_\Omega} ||h_b(\widetilde{\Box_{\bar{z}}^k B_z})||^p d\lambda(z) < \infty.$$

Proof. To show that $ii) \Rightarrow i)$, it suffices by Theorem 8.4.3 to prove that $ii)$ implies that $V_k b \in L^p(T_\Omega, d\lambda)$ and this follows easily from the inequality

$$\begin{aligned} |V_k b(z)| &= C_{n,k} |\langle h_b(\widetilde{\Box_{\bar{z}}^k B_z}), \widetilde{\Box_{\bar{z}}^k B_z} \rangle| \\ &\leq C_{n,k} ||h_b(\widetilde{\Box_{\bar{z}}^k B_z})||_{A^2} ||\widetilde{\Box_{\bar{z}}^k B_z}||_{A^2} \\ &= C_{n,k} ||h_b(\widetilde{\Box_{\bar{z}}^k B_z})||_{A^2}. \end{aligned}$$

That $i) \Rightarrow ii)$ for $2 \leq p < \infty$, follows from part i) of Lemma 8.3.2.

It remains to prove that $i) \Rightarrow ii)$ for the range $1 \leq p < 2$. Let us first show the implication for $p = 1$. By Theorem 8.4.3, it suffices to show that if $V_k b \in L^1(T_\Omega, d\lambda)$ then ii) holds. We recall that $h_b = h_{V_k b}$ and that the following representation holds:

$$h_{V_k b} = \int_{T_\Omega} V_k b(w) h_{f_w} d\lambda(w),$$

where $f_w(z) = \Delta^{2\frac{n}{r}}(\Im z)\Delta^{2\frac{n}{r}}(\Im w)\Delta^{-4\frac{n}{r}}(\frac{z - \overline{w}}{i})$ and h_{f_w} is the rank 1 Hankel operator given by

$$h_{f_w} g = \Delta^{2\frac{n}{r}}(\Im w)\Delta^{-2\frac{n}{r}}(\frac{\cdot - \overline{w}}{i})\overline{g(w)}.$$

It follows that

$$h_{f_w}(\widetilde{\Box_{\overline{z}}^k B_z}) = \Delta^{2\frac{n}{r}}(\Im w)\Delta^{-2\frac{n}{r}}(\frac{\cdot - \overline{w}}{i})\Delta^{k+\frac{n}{r}}(\Im z)\Delta^{-(k+2n/r)}(\frac{z-\overline{w}}{i}).$$

Using Lemma 8.2.1, we obtain

$$||h_{f_w}(\widetilde{\Box_{\overline{z}}^k B_z})|| = C\Delta^{n/r}(\Im w)\Delta^{k+n/r}(\Im z)|\Delta^{-(k+2n/r)}(\frac{z-\overline{w}}{i})|.$$

It follows using Lemma 8.2.1 again that

$$\begin{aligned}\int_{T_\Omega} ||h_{f_w}(\widetilde{\Box_{\overline{z}}^k B_z})||d\lambda(z) &= C\Delta^{n/r}(\Im w)\int_{\mathcal{D}} |\Delta^{-(k+2n/r)}(\frac{z-\overline{w}}{i})|\Delta^{(k+n/r)-2n/r}(\Im z)dV(z)\\ &= C_{n,k} < \infty.\end{aligned}$$

We obtain finally that

$$\begin{aligned}\int_{T_\Omega} ||h_b(\widetilde{\Box_{\overline{z}}^k B_z})||d\lambda(z) &\leq \int_{T_\Omega} |V_k b(w)|(\int_{\mathcal{D}} ||h_{f_w}(\widetilde{\Box_{\overline{z}}^k B_z})||d\lambda(z))d\lambda(w)\\ &\leq C\int_{T_\Omega} |V_k b(w)|d\lambda(w).\end{aligned}$$

Now, considering the sublinear operator $H : b \mapsto ||h_b \widetilde{\Box_{\overline{\cdot}}^k B_\cdot}||$, we have by the previous and Theorem 8.4.3 and Theorem 8.4.4 that H is bounded from $\mathbb{B}^1$ to $L^1(\mathcal{D}, d\lambda)$. We also have by Theorem 8.3.1 and Theorem 8.4.4 that H is bounded from $\mathbb{B}^2$ to $L^2(T_\Omega, d\lambda)$. We deduce by interpolation that H is bounded from $\mathbb{B}^p$ to $L^p(T_\Omega, d\lambda)$, whenever $1 \leq p \leq 2$. It follows that we have $i) \Rightarrow b \in \mathbb{B}^p \Rightarrow ii)$. The proof is complete. □

From the first part of the proof of the above theorem, we have the following.

Theorem 8.5.3. *Let $b \in L^2(T_\Omega)$ and $2 \leq p < \infty$. The following conditions are equivalent*

i) $h_b \in \mathcal{S}_p$.

ii) For every integer $k \geq 0$,

$$\int_{T_\Omega} ||h_b(\widetilde{\Box_{\overline{z}}^k B_z})||^p d\lambda(z) < \infty.$$

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